Mathematical Analysis supported by wxMaxima

Laboratory

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Project: Innovative Open Source Courses for Computer Science

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Contents

Innovative Open Source Courses for Computer Science

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Innovative Open Source Courses for Computer Science

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01. Introduction to wxMaxima

Mathematical Analysis supported by wxMaxima

01. Basic terms

- We enter commands on separate lines (input lines).

  Their execution is ensured by simultaneously pressing the `Shift` keys and `Enter` or by clicking on in the menu icon ➡ (Send the current cell to maxima).

- Input lines are listed with (%i1).

- Output lines are listed with (%o1).

- The numbers for the input line and the corresponding output line are identical and based on this number, we can refer to the content of these lines.

```
(%i1) First input line.
(%o1) First output line.
(%i2) Second input line.
(%o2) Second output line.
```

01. Basic terms

- The commands are executed on new separate lines (output lines).
- Commands on input lines can be terminated with the symbol ; or the $ symbol, which suppresses the display of the corresponding output.

```
(%i1) solve(0=x+2,x);
(%o1) [x = -2]
(%i2) %i1;
(%o2) solve(0 = x + 2, x)
(%i3) %o1;
(%o3) [x = -2]
```

01. Basic terms

- We can enter more commands on the input line, but we must separate them with symbols ; or $.
- We can also structure the command on multiple input lines.

```
(%i1) a:2;b:3;solve(a*x+b*x^2=0,x)
```
(a) 2

(b) 3

(%o1) $[x=-\frac{2}{3}, x=0]$

```
(%i2) a:2$ b:3$ solve(a*x+b*x^2=0,x);
```
(%o2) $[x=-\frac{2}{3}, x=0]$

```
(%i3) a:2$
      b:3$
      solve(a*x+b*x^2=0,x);
```
(%o3) $[x=-\frac{2}{3}, x=0]$

01. Basic terms

We can save the output in various shapes and then use it in other programs.

(%o3) $[x = -\frac{2}{3}, x = 0]$

Output (%o3) from the previous window we can:

- Copy Ctrl C and Ctrl V, respectively copy as text (can be used eg. for MSWord equation editor): `x=-2/3,x=0`,
- Copy as LaTeX `\[x=-\frac{2}{3}\operatorname{,}x=0\]`,
- Copy as MathML, image, RTF, SVG. . .

The wxMaxima environment has a well-designed user help, which can be found in the `Help` menu. You can also open Help by pressing the `F1` key.

You can also find the manual on the website `https://maxima.sourceforge.io/docs/manual/maxima_369.html`.

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02. Basic Commands

- Using `apropos` we find out the exact name of the command using part of its name.

```
(%i1) apropos("plot")
(%o1) [barsplot,boxplot,contour_plot,get_plot_option,gnuplot,...
```

- Command `describe` prints a description of the entered command.

```
(%i1) describe(plot2d)$
      -- Function:  plot2d
      plot2d (<expr><,<range_x><,<options><)
      plot2d (<expr_<>=<expr_<>,<range_x><,<range_y><,<options><)
      plot2d ([parametric,<expr_x><,<expr><_y,<range><],<options><)
      plot2d ([discrete,<points><],<options><)
      plot2d ([contour,<expr><],<range_x><,<range_y><,<options><)
      plot2d ([<type_<>,...,<type_n><],<options><)
      There are 5 types of plots that can be plotted by 'plot2d':
            1.  Explicit functions.  'plot2d' ...
      ...
```

02. Basic Commands

- Expressions are entered using the usual characters of operations, sessions and functions.
- Arguments of functions and commands are in parentheses.
- Multiplication symbol `*` must be entered!
- The exponentiation is specified by the character `^` or the pair `**`.
- Symbol `:` is used to assign a value to the right of the expression to the left.
- The following commands solve the equation $2x + 3x^2 = 0$ with unknown variable x.

```
(%i1) a:2$ b:3$ solve(a*x+b*x^2=0,x);
```

(%o1) $[x = -\frac{2}{3}, x = 0]$

- With the `kill` command we can remove variables with all their assignments and properties from memory.

```
(%i1) kill(a,b)
        /* removes all bindings from the arguments a,b */
(%i2) kill(all) /* removes all items on all infolists */
```

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02. Basic Commands

- In the menu `View` and submenu `Display equations` we can change display output lines for shapes `in 2D` (default form), `as 1D ASCII` or `as ASCII Art`.
- You can also change the output settings with the command `set_display`.

(%i1) `x/sqrt(x^2+1);set_display('none)$`

(%o1) $\frac{x}{\sqrt{x^2+1}}$ `/* in 2D */`

(%i1) `x/sqrt(x^2+1);set_display('ascii)$`

(%o1) $x/\mathrm{sqrt}(x^2+1)$ `/* as 1D ASCII */`

(%i2) `x/sqrt(x^2+1);set_display('xml)$`

```
             x
(%o2) ----------       /* as ASCII Art */
               2
      sqrt(x  + 1)
```

03. Working with Numbers and Basic Constants

- Maxima can work with real numbers written in numerical or symbolic form.
- The way of writing real numbers can be set in the menu `Numeric` using the switch `Numeric Output` between numeric and symbolic display.
- The setting of the variable `numer` determines the method of displaying.
- By default, 16 digits (including the decimal point) are displayed.
- The display accuracy is defined by the variable `fpproc` and affects the display using `bfloat`. Output `float` always displays the same.
- By default, complex numbers are entered in algebraic form (`rectform`). They can be converted to trigonometric (exponential) form using the command `polarform`.

```
(%i1) z:1+%i;
(z)   i+1
(%i2) polarform(z)+rectform(z);
```
(%o2) $\sqrt{2}\,e^{\frac{i\pi}{4}}+i+1$

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```
(%i1) z:1+%i;
```
(z) $i+1$

```
(%i2) polarform(z)+rectform(z);
```
(%o2) $\sqrt{2}\, e^{\frac{i\,\pi}{4}}+i+1$

03. Working with Numbers and Basic Constants

- We can increase or decrease the accuracy practically indefinitely.
- We can change it globally and locally for only one variable or command.

```
(%i1) log(2);
(%o1) log(2)
(%i2) log(2),numer;
(%o2) 0.693147180559453
(%i3) float(log(2));
(%o3) 0.693147180559453
(%i4) bfloat(log(2));
(%o4) 6.93147180559453b − 1
(%i5) log(2),bfloat;
(%o5) 6.93147180559453b − 1
(%i6) bfloat(log(2)),fpprec=34;
(%o6) 6.931471805599453094172321214581766b − 1
(%i7) bfloat(log(2)),fpprec=134;
(%o7) 6.93147180559945[106digits]8552023575813b − 1
```

03. Working with Numbers and Basic Constants

- Numeric constants e, π, i (imaginary unit) have the prefix `%`, i.e. `%e`, `%pi`, `%i`. They have the `%` prefix even if they are part of or the result of a calculations.
- Maxima has predefined constants `inf`, `minf` for real infinite ∞, $-\infty$.
- Maxima has predefined constants `infinity` for complex infinity.
- Logical constants `true` and `false` they represent truth and untruth.

(%i1) `%pi+%i+%e;`

(%o1) $\pi + \%i + \%e$

(%i2) `[minf,inf];`

(%o2) $[-\infty, \infty]$

(%i3) `infinity;`

(%o3) *infinity*

04. Assignments and Functions

- Maxima contains many more functions than standard programming languages.
  These are not only the functions themselves, but also various functions for their support.
- The `:` operator we use to assign values or expressions to variables.
- We define functions using the assignment `:=`.

```
(%i1) f(x):=x^2+2*x+3;
```
(%o1) $f(x) := x^2 + 2x + 3$
```
(%i6) f(x);f(y);f(x+1);
      f(-2);f(1);
```
(%o2) $x^2 + 2x + 3$
(%o3) $y^2 + 2y + 3$
(%o4) $(x+1)^2 + 2(x+1) + 3$
(%o5) 3
(%o6) 6

04. Assignments and Functions

Maxima contains many elementary functions. They are, for example:

- `exp(x)=%e^x`, `log(x)`,
- trigonometric functions, their inverse functions
  `sin(x)` and `asin(x)`, `cos(x)` and `acos(x)`, `tan(x)` and `atan(x)`, `cot(x)` and `acot(x)`,
- hyperbolic functions and their inverse functions
  `sinh(x)` and `asinh(x)`, `cosh(x)` and `acosh(x)`, `tanh(x)` and `atanh(x)`, `coth(x)` and `acoth(x)` etc.

We can use the command `print` to format the report.

```
(%i3)  a:2$ b:log(2),numer$
       print("Logarithm of a number",a,
             " is ",log(a),"=",b)$
       Logarithm of a number 2 is log(2) = 0.6931471805599453
```

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We can use the command `print` to format the report.

```
(%i3)  a:2$ b:log(2),numer$
       print("Logarithm of a number",a,
             " is ",log(a),"=",b)$
```

Logarithm of a number 2 is $\log(2) = 0.6931471805599453$

04. Assignments and Functions

The basic functions also include:

```
(%i4)  f(x):=sign(x)$ g(x):=abs(x)$
       print(f(-3.6),f(-3.2),f(-3),f(0),f(3),f(3.2),f(3.6))$
       print(g(-3.6),g(-3.2),g(-3),g(0),g(3),g(3.2),g(3.6))$
       neg neg neg zero pos pos pos
       3.6 3.2 3 0 3 3.2 3.6
(%i6)  f(x):=floor(x)$       /* bottom whole of x */
       print(f(-3.6),f(-3.2),f(-3),f(0),f(3),f(3.2),f(3.6))$
       −4 −4 −3 0 3 3 3
(%i8)  f(x):=round(x)$
            /* rounded x to the nearest integer number */
       print(f(-3.6),f(-3.2),f(-3),f(0),f(3),f(3.2),f(3.6))$
       −4 −3 −3 0 3 3 4
(%i10) f(x):=truncate(x)$
            /* removes all digits after the decimal point */
       print(f(-3.6),f(-3.2),f(-3),f(0),f(3),f(3.2),f(3.6))$
       −3 −3 −3 0 3 3 3
(%i12) f(x):=ceiling(x)$     /* upper integer x */
       print(f(-3.6),f(-3.2),f(-3),f(0),f(3),f(3.2),f(3.6))$
       −3 −3 −3 0 3 4 4
```

04. Assignments and Functions

- Maxima also includes many features to support them.
- Some of them are not implemented directly in the wxMaxima environment, but in external libraries called packages.
- These packages are loaded into the system using the `load` command.
- We will show the `spangl` package for an example to support work with trigonometric functions.

(%i2)
```
print(tan(%pi/8), ratsimp(tan(%pi/8)),
        trigsimp(tan(%pi/8)))$
```

$$\tan\left(\frac{\pi}{8}\right) \ \tan\left(\frac{\pi}{8}\right) \ \frac{\sin\left(\frac{\pi}{8}\right)}{\cos\left(\frac{\pi}{8}\right)}$$

(%i3)
```
load(spangl);
```

(%o3) *../share/trigonometry/spangl.mac*

(%i4)
```
tan(%pi/8);
```

(%o4) $\sqrt{2}-1$

05. Working with Expressions

Many times we need to change the conditions only locally for a particular calculation without to change global settings. For this purpose, Maxima has a very efficient `ev` command.

- The command `ev` allows defining a specific environment within a single command.
- After entering the command `ev(a,b1,b2,..., bn)` the expression `a` is evaluated if the conditions `b1`, `b2`, ..., `bn` are met.
- These conditions can be equations, assignments, functions, switches (logical settings).

The example shows an example of solving a quadratic equation using the command `solve`.

- Variables `a`, `b`, `c` after executing the command `ev` they do not have values assigned.

(%i1) `ev(solve(a*x^2+b*x+c=0,x),a:2,b:-1,c=-3);`

(%o1) $[x=\frac{3}{2}, x=-1]$

(%i2) `solve(a*x^2+b*x+c=0,x);`

(%o2) $[x=-\frac{\sqrt{b^2-4ac}+b}{2a}, x=\frac{\sqrt{b^2-4ac}-b}{2a}]$

05. Working with Expressions

Maxima offers several commands for simplifying and editing various expressions.

- The basic functions can be found in the `Simplify` menu.
- Maxima offers using the `example` command examples of individual commands.
- Let's take a look at some of the examples offered by `example(ratsimp)`.

(%i2) `f(x):=b*(a/b-x)+b*x+a$`

`print(f(x),"?",ratsimp(f(x)))$`

$bx + b(\frac{a}{b} - x) + a \ ? \ 2a$

(%i3) `ratsimp(a+1/a);`

(%o3) $\frac{a^2+1}{a}$

(%i4) `ev(x^(a+1/a),ratsimp);`

(%o4) $x^{a+\frac{1}{a}}$

(%i5) `ev(x^(a+1/a),ratsimpexpons);`

(%o5) $x^{\frac{a^2+1}{a}}$

05. Working with Expressions

- Function `expand` multiplies the relevant members in the expression.
  Function `factor` on the contrary, it decomposes the expression.
  Function `gfactor` it does so over a field of complex numbers.

(%i1) `f(x):=(x+1)*(x^2-4)*(x^2+4)$`
(%i3) `ratsimp(f(x));expand(f(x));`
(%o2) $x^5+x^4-16x-16$
(%o3) $x^5+x^4-16x-16$
(%i6) `factor(f(x));gfactor(f(x));factor(100);`
(%o4) $(x-2)(x+1)(x+2)(x^2+4)$
(%o5) $(x-2)(x+1)(x+2)(x-2\%i)(x+2\%i)$
(%o6) $2^2 5^2$

- We decompose a rational polynomial function into partial fractions using the `partfrac`.

(%i1) `partfrac((x+1)/(x^2-2*x+1),x);`
(%o1) $\frac{1}{x-1}+\frac{2}{(x-1)^2}$

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  Function `factor` on the contrary, it decomposes the expression.
  Function `gfactor` it does so over a field of complex numbers.

```
(%i1) f(x):=(x+1)*(x^2-4)*(x^2+4)$
(%i3) ratsimp(f(x));expand(f(x));
```
(%o2) $x^5 + x^4 - 16x - 16$
(%o3) $x^5 + x^4 - 16x - 16$
```
(%i6) factor(f(x));gfactor(f(x));factor(100);
```
(%o4) $(x-2)(x+1)(x+2)(x^2+4)$
(%o5) $(x-2)(x+1)(x+2)(x-2\%i)(x+2\%i)$
(%o6) $2^2 5^2$

- We decompose a rational polynomial function into partial fractions using the `partfrac`.

```
(%i1) partfrac((x+1)/(x^2-2*x+1),x);
```
(%o1) $\frac{1}{x-1} + \frac{2}{(x-1)^2}$

05. Working with Expressions

We can substitute expressions using the commands `subst(a,b,c)` and `ratsubst(a,b,c)`.

- The expression `a` will be replaced by `b` and subsequently substituted into the expression `c`.
- When using the `subst` command must be `b` the simplest part (atom) or a complete subexpression of the expression `c`.
- In the example, the subexpression is not `x+y` complete (missing `z`).
- The `ratsubst` command it also modifies the resulting expression.

(%i2) `subst(x+y,a,a^2+b^2); ratsubst(x+y,a,a^2+b^2);`

(%o1) $(y+x)^2+b^2$

(%o2) $y^2+2xy+x^2+b^2$

(%i4) `subst(a,x+y,x+y+z); ratsubst(a,x+y,x+y+z);`

(%o3) $z+y+x$

(%o4) $z+a$

06. Limits and Derivatives

In the `Calculus` menu we find functions for solving basic problems of mathematical analysis (limits, derivation, integration, sums of series, calculate products, ...).

We calculate the limits using the command `limit`.

- The last parameter determines the direction of unilateral limits, has the values `plus` or `minus` and is optional.

  If not specified, Maxima calculates the limit as complex.

- With the command `limit(f(x),x,a)` we calculate the limit $\lim\limits_{x\to a} f(x)$.
- With the command `limit(f(x),x,a,plus)` we calculate the limit $\lim\limits_{x\to a^+} f(x)$.

```
(%i4) limit(1/x,x,0);       limit(1/x,x,0,plus);
            limit(1/x,x,0,minus);  limit(1/x,t,0);
```

(%o1) *infinity*

(%o2) ∞

(%o3) $-\infty$

(%o4) $\frac{1}{x}$

06. Limits and Derivatives

If we use an apostrophe ' before a command,
this command will not be executed, the command will only be displayed.

```
(%i2) limit(((1-n)/(1+3*n))^(1+4*n),n,inf);
      'limit(((1-n)/(1+3*n))^(1+4*n),n,inf);
```

(%o1) 0

(%o2) $\lim\limits_{n\to\infty}\left(\frac{1-n}{3n+1}\right)^{4n+1}$

```
(%i2) 2+3; '2+3;
```

(%o1) 5

(%o2) 5

```
(%i4) solve(x+1=0,x);'solve(x+1=0,x);
```

(%o3) $[x=-1]$

(%o4) $solve(x+1=0,x)$

06. Limits and Derivatives

Derivatives are calculated using the command `diff`.

The parameter that determines the order of derivation is optional.

```
(%i4) f(x):=2*x^4-3*x+sin(x);
      print("f'=",diff(f(x),x),
            "=",diff(f(x),x,1))$
      print("f''=",diff(diff(f(x),x),x),
            "=",diff(f(x),x,2),
            "=",diff(f(x),x,1,x,1))$
      print("f^(10)=",diff(f(x),x,10),
            "=",diff(f(x),x,1,x,9))$
```

(%o1) $f(x) := 2x^4 - 3x + \sin(x)$

$f' = \cos(x) + 8x^3 - 3 = \cos(x) + 8x^3 - 3$

$f'' = 24x^2 - \sin(x) = 24x^2 - \sin(x) = 24x^2 - \sin(x)$

$f\hat{}(10) = -\sin(x) = -\sin(x)$

06. Limits and Derivatives

We calculate partial derivatives using the same command `diff`.

```
(%i3) g(x,y):=x^3*y^2-1;
      print("g'_x=",diff(g(x,y),x),
            ", respectively
              g'_y=",diff(g(x,y),y,1))$
      print( "g''_(xx)=",diff(g(x,y),x,2),
            ", g''_(yx)=",diff(g(x,y),y,1,x,1),
            ", g''_(xy)=",diff(g(x,y),x,1,y,1),
            ", g''_(yy)=",diff(g(x,y),y,1,y,1))$
```

(%o1) $g(x,y) := x^3y^2 - 1$

$g'\_x = 3x^2y^2$, respectively $g'\_y = 2x^3y$

$g''\_(xx) = 6xy^2$, $g''\_(yx) = 6x^2y$, $g''\_(xy) = 6x^2y$, $g''\_(yx) = 2x^3$

06. Limits and Derivatives

We calculate the Taylor polynomial nth degree using the command `taylor`.

- You can find this command in the Calculus menu and the Get Series... submenu.
- We calculate Taylor series of functions f degree n in the middle c with the command `taylor(f(x),x,c,n)`.
- Its coefficients are obtained using the command `coeff`.
- The use of this command depends on the `taylor` command.

```
(%i1) t1:taylor(sin(x),x,0,5); t2:taylor(sin(x),x,-1,4);
```

(t1) $x-\frac{x^3}{6}+\frac{x^5}{120}+\cdots$

(t2) $-\sin(1)+\cos(1)(x+1)+\frac{\sin(1)(x+1)^2}{2}-\frac{\cos(1)(x+1)^3}{6}-\frac{\sin(1)(x+1)^4}{24}+\cdots$

```
(%i3) print(coeff(sin(x),x,5)," and ",coeff(t1,x,5),
        " and ",coeff(t2,x,5))$
```

0 and $\frac{1}{120}$ and $\frac{\cos(1)}{120}$

06. Limits and Derivatives

In the example, the Taylor polynomial of a given polynomial is calculated in another way.

Command `taylor` gives three points at the end, even if development is closed.

```
(%i1) f(x):=2*x^5-x^4-3*x^3-x+1;
```
(%o1) $f(x) := 2x^5 - x^4 + (-3)x^3 - x + 1$
```
(%i2) tp1:taylor(f(x),x,-1,5);
```
(tp1) $2 + 4(x+1) - 17(x+1)^2 + 21(x+1)^3 - 11(x+1)^4 + 2(x+1)^5 + \cdots$
```
(%i4) ratsimp(tp1);expand(tp1);
```
(%o3) $2x^5 - x^4 - 3x^3 - x + 1$

(%o4) $2x^5 - x^4 - 3x^3 - x + 1$
```
(%i6) tpx:ratsubst(t,x+1,f(x));subst(x+1,t,tpx);
```
(tpx) $2t^5 - 11t^4 + 21t^3 - 17t^2 + 4t + 2$

(tp2) $2(x+1)^5 - 11(x+1)^4 + 21(x+1)^3 - 17(x+1)^2 + 4(x+1) + 2$
```
(%i7) tp1-tp2;
```
(%o7) $0 + \cdots$

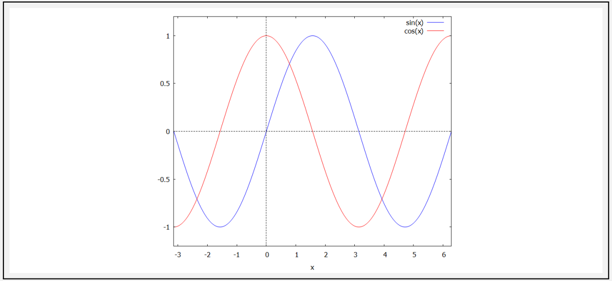
07. Function graphs

We can plot the function graph in several ways.

- The easiest way is to choose `Plot` in the menu submenu `Plot 2d ...`.
- If we choose `Format=gnuplot`, the function is rendered by the command `plot2d` using the Open Source program `Gnuplot` to a new window.

  `Gnuplot` is automatically installed together with Maxima.

```
(%i1) plot2d([sin(x),cos(x)],[x,-%pi,2*%pi],[y,-1.2,1.2],
        [plot_format, gnuplot])$
```

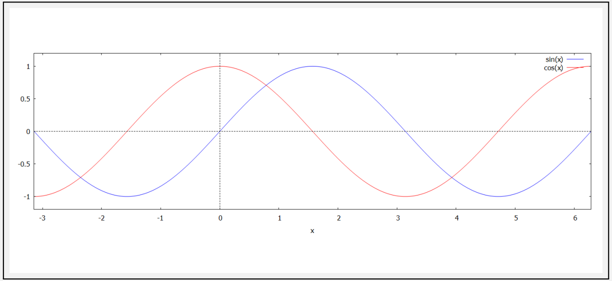


07. Function graphs

The graphs of functions were not displayed in the real ratio of the *x* and *y* axes, but were optimized for the screen.

- We can use e.g. `same_xy` parameter for proper display.

```
(%i1) plot2d([sin(x),cos(x)],[x,-%pi,2*%pi],[y,-1.2,1.2],
          [plot_format, gnuplot],[same_xy])$
```

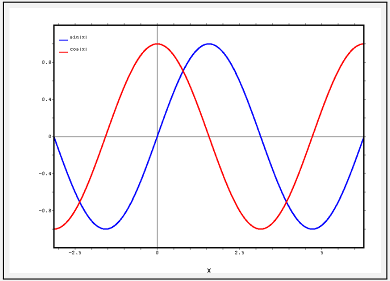


07. Function graphs

If we choose `Format=wxmaxima`:

- Maxima will plot the graph using the command `plot2d` to a new window.
- We can only save the image in postscript.

```
(%i1) plot2d([sin(x),cos(x)],[x,-%pi,2*%pi],[y,-1.2,1.2],
        [plot_format, xmaxima])$
```



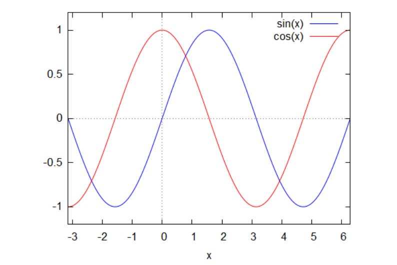
07. Function graphs

If we choose `Format=inline`:

- Maxima draws a graph using the command `wxplot2d` into your environment.

```
(%i1) wxplot2d([sin(x),cos(x)],[x,-%pi,2*%pi],
            [y,-1.2,1.2])$
```

(%o1)



Commands `plot2d` and `wxplot2d` they have the same syntax and have many more parameters.

- Parameters can be found, for example, with the command `describe(plot2d)`.

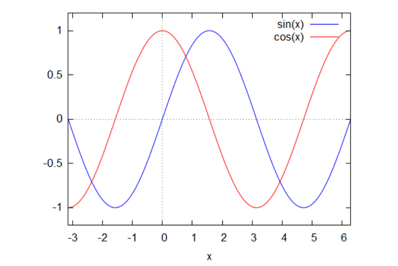
07. Function graphs

If we choose `Format=inline`:

- Maxima draws a graph using the command `wxplot2d` into your environment.

```
(%i1) wxplot2d([sin(x),cos(x)],[x,-%pi,2*%pi],
           [y,-1.2,1.2])$
```

(%o1)



Commands `plot2d` and `wxplot2d` they have the same syntax and have many more parameters.

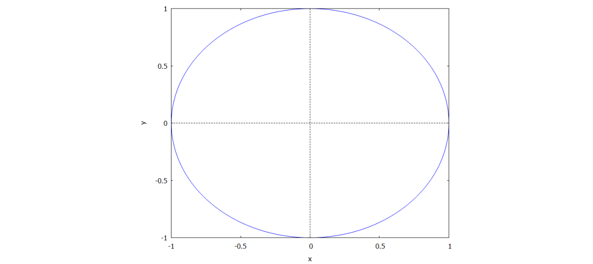
- Parameters can be found, for example, with the command `describe(plot2d)`.

07. Function graphs

If we want to display a implicit function, we need to load it `implicit_plot` library.

- In newer versions (at least wxMaxima 21.05.2) it is no longer necessary.

```
(%i1) load(implicit_plot);
(%o1) ../share/contrib/implicit_plot.lisp
(%i2) implicit_plot(x^2+y^2-1, [x,-1,1], [y,-1,1])$
      implicit_plot is now obsolete.  Using plot2d instead:
      plot2d (y^2+x^2-1=0, [x,-1,1], [y,-1,1])
```



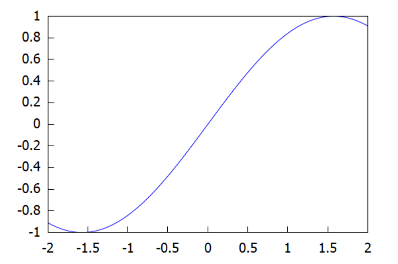
07. Function graphs

We can plot the function graph in several ways.

- It is better to use `wxdraw2d` or `draw2d` commands and direct the output to `Gnuplot`.
- These commands have a slightly different syntax than the `wxplot2d`, `plot2d`.
  The print parameters are simpler and clearer.
- The plotted function must be in the command `explicit`, `parametric` or `implicit`.

(%i1) `wxdraw2d(explicit((sin(x)),x,-2,2))$`

(%o1)

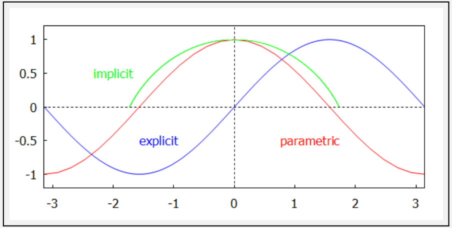


07. Function graphs

Plotting with `wxdraw2d` and `draw2d` commands.

- We can use e.g. `proportional_axes` parameter for proper display.

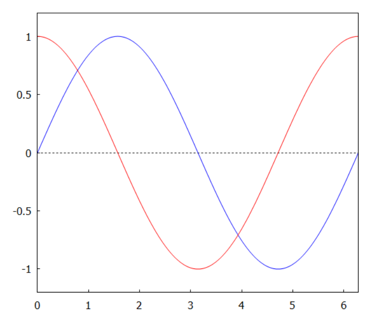
```
(%i1) draw2d(proportional_axes=xy,xaxis=true,yaxis=true,
      xrange=[-%pi,%pi],yrange=[-1.2,1.2],
      color=blue,explicit((sin(x)),x,-%pi,%pi),
      label(["explicit",-1.25,-.5]),
      color=red,parametric(t,cos(t),t,-%pi,%pi),
      label(["parametric",1.25,-.5]),
      color=green,implicit(x^2+(y+1)^2-4,x,-2,2,y,0,1),
      label(["implicit",-2,.5]))$
```



07. Function graphs

- `draw2d` command.

```
(%i1) draw2d(xaxis=true,yaxis=true,
      xrange=[0,2*%pi],yrange=[-1.2,1.2],
      color=blue,explicit((sin(x)),x,0,2*%pi),
      color=red,explicit((cos(x)),x,0,2*%pi))$
```

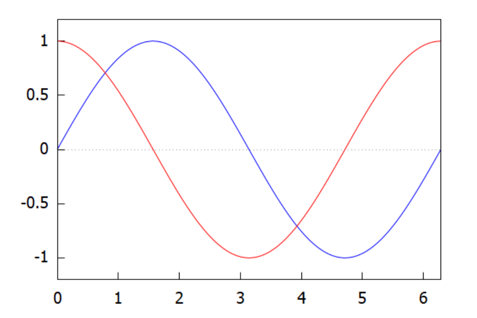


07. Function graphs

- `wxdraw2d` command.

(%i1)
```
wxdraw2d(xaxis=true,yaxis=true,
xrange=[0,2*%pi],yrange=[-1.2,1.2],
color=blue,explicit((sin(x)),x,0,2*%pi),
color=red,explicit((cos(x)),x,0,2*%pi))$
```

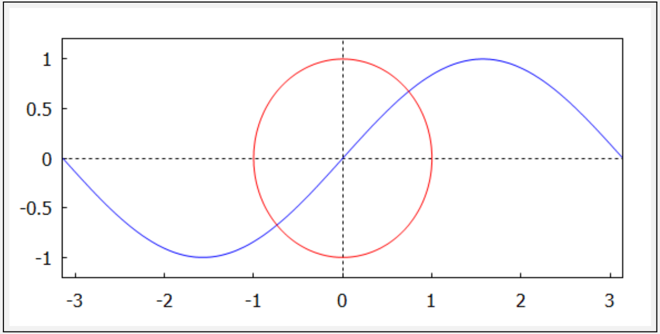
(%o1)



07. Function graphs

- We plot a parametric curve or function in a similar way.

```
(%i1) draw2d(proportional_axes=xy,xaxis=true,yaxis=true,
      xrange=[-%pi,%pi],yrange=[-1.2,1.2],
      color=blue,explicit((sin(x)),x,-%pi,%pi),
      color=red,nticks=300,
      parametric(cos(t),sin(t),t,0,2*%pi))$
```



08. Sequences and Series

Sequences can be created in Maxima in several ways.

- Sequences can be created, for example, using the command `makelist` or with the statements of the cycle `for..do`.
- Command `makelist` creates a list that we can display as a whole and by members.

```
(%i2)  S1:makelist(2*n^2-1,n,1,10);
       S2:makelist(2*n^2-1,n,2,10,2);
(S1)   [1,7,17,31,49,71,97,127,161,199]
(S2)   [7,31,71,127,199]
(%i4)  S1[1];S2[1];S1[10];
(%o3)  1
(%o4)  7
(%o5)  199
(%i6)  S1[12];
       inpart: invalid index 12 of list or matrix.
       -- an error. To debug this try: debugmode(true);
```

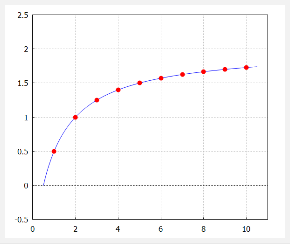
08. Sequences and Series

- The sequence is generated with its patterns and then plotted using the `draw2d` command.
- Arranged pairs are enclosed in square brackets and can be displayed as points in a plane.

(%i1)
```
S1:makelist([n,(2*n-1)/(n+1)],n,1,10);
```

(S1) $[[1, \frac{1}{2}], [2, 1], [3, \frac{5}{4}], [4, \frac{7}{5}], [5, \frac{3}{2}], [6, \frac{11}{7}], [7, \frac{13}{8}], [8, \frac{5}{3}], [9, \frac{17}{10}], [10, \frac{19}{11}]]$

(%i2)
```
draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[0,11],yrange=[-0.5,2.5],
color=blue,explicit((2*n-1)/(n+1),n,0.5,10.5),
point_type=7,color=red,points(S1))$
```



08. Sequences and Series

- Using the command `for..do` we will list several members of the sequence $\{2n^2 - 1\}_{n=1}^{\infty}$.

```
(%i1) (for n:1 thru 15 do (a_n: 2*n^2-1, print(a_n)) )$
      1
      7
      17
      31
      49
      71
      97
      127
      161
      199
      241
      287
      337
      391
      449
```

08. Sequences and Series

- A nice example of using the command `for..do` is a Fibonacci sequence.

```
(%i3) a0:0$ a1:1$ (for i:1 thru 14
                do (an:a1+a0,print(an),a1:a0,a0:an))$
      1
      1
      2
      3
      5
      8
      13
      21
      34
      55
      89
      144
      233
      377
```

08. Sequences and Series

We can calculate the sum of the series with the `sum` command.
This command can be found in the menu `Calculus` and the `Calculate Sum...` submenu.

- We calculate the finite and infinite sum using the command `sum`.

```
(%i1) sum(2*n^2-1,n,1,8);
(%o1) 400
```

- Maxima can calculate the exact sum of some infinite series.

```
(%i2) sum(1/k^2,k,1,inf);
      sum(1/k^2,k,1,inf),simpsum;
```

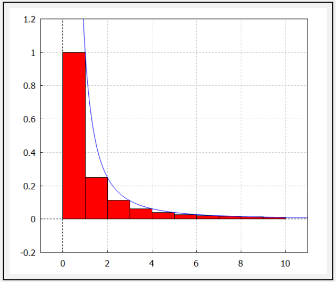
(%o1) $\sum_{k=1}^{\infty} \left(\frac{1}{k^2}\right)$

(%o2) $\frac{\pi^2}{6}$

08. Sequences and Series

- The number series $\sum_{k=1}^{\infty} \left(\frac{1}{k^2}\right)$ can be graphically represented as follows.

```
(%i1) a(n):=1/n^2$
      rec:makelist(rectangle([i-1,0],[i,a(i)]),i,1,10)$
      draw2d(grid=true,xaxis=true,yaxis=true,
      xrange=[-1,11],yrange=[-0.2,1.2],
      border=true,color=black,fill_color=red,rec,
      color=blue,explicit(a(n),n,0,11))$
```



02. Real functions

Mathematical Analysis supported by wxMaxima

01. Basic terms

- **Binary relation** f between the sets $A \neq \emptyset$ and $B \neq \emptyset$ is every $f \subset A \times B$.
- If for each $x \in A$ there is at most one $y \in B$ such that $[x; y] \in f$, then relation f is called **function** (**map**, **mapping**, **transformation**) from set A to set B, label $f\colon A \to B$.
  We also write as $[x; y] \in f$ or $y = f(x)$.
- $x \in A$ Pattern, independent variable, input value, argument.
- $y \in B$ Image, dependent variable, output value, value of the function.
- $D(f) = \{x \in A, \exists y \in B\colon [x; y] \in f\}$ Domain of the function f (set of patterns).
- $H(f) = \{y \in B, \exists x \in D(f)\colon [x; y] \in f\}$ Codomain of values of the function f (set of images).
- Relations and functions are sets of ordered pairs.
- $f = g$ represents the equivalence of $[x; y] \in f \Leftrightarrow [x; y] \in g$,
  i.e. $D(f) = D(g)$ and $f(x) = g(x)$ for all $x \in D(f)$.

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02. Sequences (of real numbers)

$\{a_n\}_{n=1}^{\infty} = \{2n-1\}_{n=1}^{\infty} = \{1, 3, 5, \ldots\}.$

- Explicit entry: $a_n = 2n - 1,\ n \in N.$
- Recurring entry: $a_1 = 1,\ a_{n+1} = a_n + 2,\ n \in N.$

```
(%i3) a(n):=2*n-1$ S:makelist(a(n),n,1,8);
(S)   [1, 3, 5, 7, 9, 11, 13, 15]
(%i4) an:1$ (for n:1 thru 8 do (print(an),an:an+2))$
      1
      3
      5
      7
      9
      11
      13
      15
```

02. Sequences (of real numbers)

A sequence $\{a_n\}_{n=1}^{\infty}$, $a_n \in R$.

- If $\{k_n\}_{n=1}^{\infty} \subset N$ is an increasing sequence (of natural numbers, indices), then $\{a_{k_n}\}_{n=1}^{\infty}$ is called **subsequence** (**selected sequence**) from $\{a_n\}_{n=1}^{\infty}$.

Subsequences of the $\{a_n\}_{n=1}^{\infty} = \{2n-1\}_{n=1}^{\infty} = \{1, 3, 5, 7, 9, 11, 13, \ldots\}$ are for example:

- $\{a_{k_n}\}_{n=1}^{\infty} = \{a_{2n}\}_{n=1}^{\infty} = \{a_2, a_4, a_6, \ldots\} = \{3, 7, 11, \ldots\} = \{4n-1\}_{n=1}^{\infty}$.
- $\{a_n\}_{n=1}^{\infty} = \{2n-1\}_{n=1}^{\infty}$.
- $\{a_n\}_{n=2}^{\infty} = \{2n-1\}_{n=2}^{\infty}$.
- $\{101, 109, 235, 637, \ldots\}$.

```
(%i2) a(n):=2*n-1$ makelist(a(n),n,1,7);
(%o2) [1,3,5,7,9,11,13]
(%i3) makelist(a(2*n),n,1,7);
(%o3) [3,7,11,15,19,23,27]
(%i4) makelist(a(2*n),n,2,7);
(%o4) [7,11,15,19,23,27]
(%i5) print(a(51),a(55),a(118),a(319))$
      101 109 235 637
```

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```
(%i2) a(n):=2*n-1$ makelist(a(n),n,1,7);
(%o2) [1, 3, 5, 7, 9, 11, 13]
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(%o3) [3, 7, 11, 15, 19, 23, 27]
(%i4) makelist(a(2*n),n,2,7);
(%o4) [7, 11, 15, 19, 23, 27]
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      101 109 235 637
```

02. Sequences (of real numbers)

- $\lim\limits_{n\to\infty} \frac{n^2+n}{n^3-2} = \lim\limits_{n\to\infty} \frac{n^3(n^{-1}+n^{-2})}{n^3(1-2n^{-3})} = \lim\limits_{n\to\infty} \frac{n^{-1}+n^{-2}}{1-2n^{-3}} = \frac{0+0}{1-0} = 0.$

```
(%i1) a(n):=(n^2+n)/(n^3-2)$
          Sa:makelist([n,a(n)],n,1,15)$
      print("limit a(n)=",limit(a(n),n,inf))$
      limit a(n)= 0
```

- $\lim\limits_{n\to\infty} \frac{n^3-2}{n^2+n} = \lim\limits_{n\to\infty} \frac{n^2(n-2n^{-2})}{n^2(1+n^{-1})} = \lim\limits_{n\to\infty} \frac{n-2n^{-2}}{1+n^{-1}} = \frac{\infty-0}{1+0} = \infty.$

```
(%i1) b(n):=(n^3-2)/(n^2+n)$
          Sb:makelist([n,b(n)],n,1,15)$
      print("limit b(n)=",limit(b(n),n,inf))$
      limit b(n)= ∞
```

02. Sequences (of real numbers)

- $\lim_{n\to\infty} \frac{n^2+n}{n^3-2} = \lim_{n\to\infty} \frac{n^3(n^{-1}+n^{-2})}{n^3(1-2n^{-3})} = \lim_{n\to\infty} \frac{n^{-1}+n^{-2}}{1-2n^{-3}} = \frac{0+0}{1-0} = 0.$

```
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```
(%i1) b(n):=(n^3-2)/(n^2+n)$
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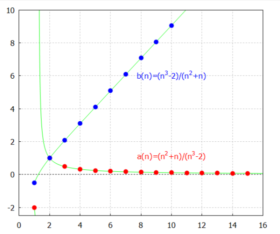
      print("limit b(n)=",limit(b(n),n,inf))$
      limit b(n)= ∞
```

02. Sequences (of real numbers)

(%i1)
```
a(n):=(n^2+n)/(n^3-2)$ Sa:makelist([n,a(n)],n,1,15)$
b(n):=(n^3-2)/(n^2+n)$ Sb:makelist([n,b(n)],n,1,15)$
draw2d(grid=true,xaxis=true,yaxis=true,
     xrange=[0,16],yrange=[-2.5,10],
color=green,explicit(a(n),n,1,16),point_type=7,
     color=red,points(Sa),
  label(["a(n)=(n^2+n)/(n^3-2)",10,a(10)+1]),
color=green,explicit(b(n),n,1,16),point_type=7,
     color=blue,points(Sb),
  label(["b(n)=(n^3-2)/(n^2+n)",10,6]))$
```



03. Number series

If $\{a_n\}_{n=1}^{\infty}$ is a sequence,

then $\sum\limits_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$ is called **(infinite number) series**.

- Number series are closely related to sequences and generalize the concept additions to an infinite number of addends. A simple example is fractions and periodic numbers.

$$\sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + a_3 + \cdots + a_k}_{s_k = \sum_{i=1}^{k} a_i \ (k\text{-th partial sum})} + \underbrace{a_{k+1} + a_{k+2} + a_{k+3} + \cdots}_{r_k = \sum_{i=k+1}^{\infty} a_i \ (k\text{-th rest})}$$

- $\{s_k\}_{k=1}^{\infty} = \{s_1, s_2, s_3, \ldots\} = \{s_n\}_{n=1}^{\infty}$ The sequence of partial sums of the series $\sum\limits_{n=1}^{\infty} a_n$.
- The relationship between $\sum\limits_{n=1}^{\infty} a_n$ and the sequence $\{s_n\}_{n=1}^{\infty}$ is mutually unambiguous.
  - $s_1 = a_1 = s_0 + a_1$.
  - $s_2 = a_1 + a_2 = s_1 + a_2$.
  - $s_3 = a_1 + a_2 + a_3 = s_2 + a_3$.

    $\ldots$
  - $s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n = s_{n-1} + a_n$.
  - $a_1 = s_1 - s_0$, where $s_0 = 0$.
  - $a_2 = s_2 - s_1$.
  - $a_3 = s_3 - s_2$.
  - $a_n = s_n - s_{n-1}$, $n \in N$.

03. Number series

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  - $s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n = s_{n-1} + a_n$.
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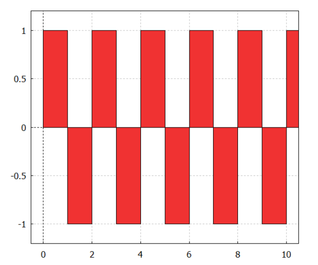
  - $a_2 = s_2 - s_1$.
  - $a_3 = s_3 - s_2$.
  - $a_n = s_n - s_{n-1}$, $n \in N$.

03. Number series

Series $\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots.$

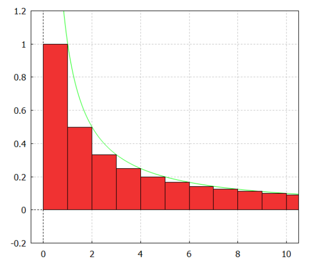
(%i1)
```
a(n):=(-1)^(n+1)$
rec:makelist(rectangle([i-1,0],[i,a(i)]),i,1,11)$
draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[-.5,10.5],yrange=[-1.2,1.2],
border=true,color=black,fill_color=red,rec)$
```



03. Number series

The Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty.$

```
(%i1)  a(n):=1/n$
       rec:makelist(rectangle([i-1,0],[i,a(i)]),i,1,11)$
       draw2d(grid=true,xaxis=true,yaxis=true,
       xrange=[-.5,10.5],yrange=[-.2,1.2],
       color=green,explicit(a(n),n,.5,11),
       border=true,color=black,fill_color=light_red,rec)$
```



03. Number series

```
(%i4) sq(q):=sum(q^n,n,1,inf)$ sq(1/2),simpsum;
      sq(1/3),simpsum; sq(-1/2),simpsum; sq(2),simpsum;
```

(%o1) 1

(%o2) $\frac{1}{2}$

(%o3) $-\frac{1}{3}$

(%o4)
```
sum:  sum is divergent.
#0:  sq(q=2) -- an error.  To debug this try:  debugmode(true);
```

- It is enough to change the value of q at the beginning in the following example.

```
(%i1) q:0.8$ a(n,q):=q^n$ peca:makelist([i,a(i,q)],i,1,11)$
      reca:makelist(rectangle([i-1,0],[i,a(i,q)]),i,1,11)$
      draw2d(grid=true,xaxis=true,yaxis=true,
      xrange=[-.5,10.5],yrange=[-4,4],
      border=true,color=black,fill_color=light_red,reca,
      label([concat("q=",string(q)),3,3.5]),
      color=blue,explicit(a(n,q),n,1,11),
      point_type=7,color=blue,points(peca))$
```

03. Number series

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(%i4) sq(q):=sum(q^n,n,1,inf)$ sq(1/2),simpsum;
      sq(1/3),simpsum; sq(-1/2),simpsum; sq(2),simpsum;
```

(%o1) 1

(%o2) $\frac{1}{2}$

(%o3) $-\frac{1}{3}$

(%o4) `sum: sum is divergent.`

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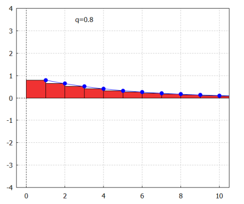
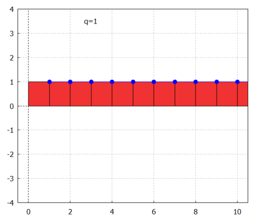
      xrange=[-.5,10.5],yrange=[-4,4],
      border=true,color=black,fill_color=light_red,reca,
      label([concat("q=",string(q)),3,3.5]),
      color=blue,explicit(a(n,q),n,1,11),
      point_type=7,color=blue,points(peca))$
```

03. Number series

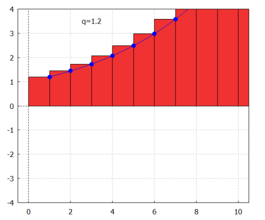
The commands will display the following graphs:



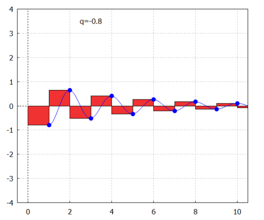
$q = 0.8$



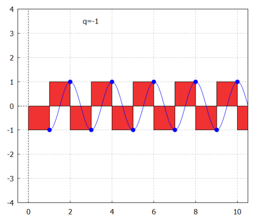
$q = 1$



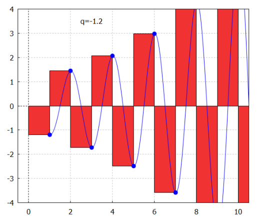
$q = 1.2$



$q = -0.8$



$q = -1$



$q = -1.2$

03. Number series

The series $\sum_{n=1}^{\infty} a_n$, $a_n \geq 0$, $n \in N$ (**non-negative terms**) always has a sum $0 \leq s = \sum_{n=1}^{\infty} a_n \leq \infty$.

Direct comparison test. $0 \leq a_n \leq b_n$, $n \in N$.

- $\sum_{n=1}^{\infty} b_n \longrightarrow .$ $\Rightarrow$ • $\sum_{n=1}^{\infty} a_n \longrightarrow .$
- $\sum_{n=1}^{\infty} a_n \longrightarrow \infty.$ $\Rightarrow$ • $\sum_{n=1}^{\infty} b_n \longrightarrow \infty.$

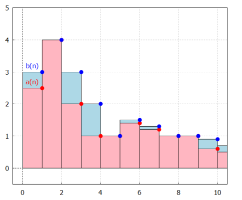
Comparison test (limit form). $0 \leq a_n \leq b_n$, $n \in N$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = p$, $0 < p < \infty$.

- $\sum_{n=1}^{\infty} a_n \longrightarrow .$ $\Leftrightarrow$ • $\sum_{n=1}^{\infty} b_n \longrightarrow .$
- $\sum_{n=1}^{\infty} a_n \longrightarrow \infty.$ $\Leftrightarrow$ • $\sum_{n=1}^{\infty} b_n \longrightarrow \infty.$

03. Number series

(%i1)
```
a:[2.5,4,2,1,1,1.4,1.2,1,1,0.6,0.5]$
pa:makelist([i,a[i]],i,1,11)$
ra:makelist(rectangle([i-1,0],[i,a[i]]),i,1,11)$
b:[3.0,4,3,2,1,1.5,1.3,1,1,0.9,0.7]$
pb:makelist([i,b[i]],i,1,11)$
rb:makelist(rectangle([i-1,0],[i,b[i]]),i,1,11)$
draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[-.5,10.5],yrange=[-.5,5],
border=true,color=black,fill_color=light_blue,
rb,color=black,fill_color=light_pink,ra,
point_type=7,color=red,points(pa),point_type=7,
color=blue,points(pb),color=red,label(["a(n)",.5,2.7]),
color=blue,label(["b(n)",.5,3.2]))$
```



03. Number series

Ratio test (d'Alembert's ratio test). $a_n > 0,\ n \in N.$

- $\frac{a_{n+1}}{a_n} \leq q < 1,\ n \in N$, where $q \in (0;1)$. $\Rightarrow$ • $\sum_{n=1}^{\infty} a_n \longrightarrow .$
- $1 \leq \frac{a_{n+1}}{a_n},\ n \in N.$ $\Rightarrow$ • $\sum_{n=1}^{\infty} a_n \longrightarrow \infty.$

d'Alembert's ratio test (limit form). $a_n > 0,\ n \in N.$

$\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = p.$

- $p < 1.$ $\Rightarrow$ • $\sum_{n=1}^{\infty} a_n \longrightarrow .$
- $p > 1.$ $\Rightarrow$ • $\sum_{n=1}^{\infty} a_n \not\longrightarrow .$

For $p = 1$ we cannot decide.

- $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n \longrightarrow \infty$, but $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{n+1}{n} = 1.$
- $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \longrightarrow$, but $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim\limits_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim\limits_{n\to\infty} \frac{n^2}{n^2+2n+1} = 1.$

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03. Number series

Root test (Cauchy root test). $a_n \geq 0,\ n \in N.$

- $\sqrt[n]{a_n} \leq q < 1,\ n \in N$, where $q \in (0;1)$. $\Rightarrow$ • $\sum_{n=1}^{\infty} a_n \longrightarrow.$
- $1 \leq \sqrt[n]{a_n},\ n \in N.$ $\Rightarrow$ • $\sum_{n=1}^{\infty} a_n \longrightarrow \infty.$

Cauchy root test (limit form). $a_n \geq 0,\ n \in N.$

$\lim_{n\to\infty} \sqrt[n]{a_n} = p.$

- $p < 1.$ $\Rightarrow$ • $\sum_{n=1}^{\infty} a_n \longrightarrow.$
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03. Number series

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03. Number series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{a^n}{n!} \longrightarrow \text{for } a > 0.$$

d'Alembert's ratio test:

- $\lim\limits_{n\to\infty} \frac{a^{n+1}}{(n+1)!} \frac{n!}{a^n} = \lim\limits_{n\to\infty} \frac{a}{n+1} = \frac{a}{\infty} = 0 < 1.$ $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} \frac{a^n}{n!} \longrightarrow$ for $a > 0$.

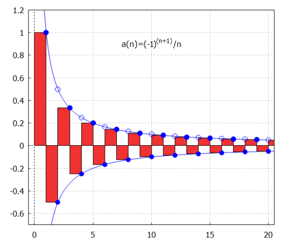
Cauchy root test:

- $\lim\limits_{n\to\infty} \sqrt[n]{\frac{a^n}{n!}} = \lim\limits_{n\to\infty} \frac{a}{\sqrt[n]{n!}} = \frac{a}{\infty} = 0 < 1.$ $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} \frac{a^n}{n!} \longrightarrow$ for $a > 0$.

```
(%i5) an(n,a):=a^n/n!$ a:2$ limit(an(n,a),n,inf,plus);
          limit(an(n+1,a)/an(n,a),n,inf,plus);
          limit((an(n,a))^(1/n),n,inf,plus);
(%o3) 0
(%o4) 0
(%o5) 0
```

03. Number series

```
(%i1) a(n):=(-1)^(n+1)/n$ pa:makelist([i,a(i)],i,1,21)$
      ra:makelist(rectangle([i-1,0],[i,a(i)]),i,1,21)$
      draw2d(grid=true,xaxis=true,yaxis=true,
      xrange=[-.5,20.5],yrange=[-.7,1.2],
      color=blue,explicit(abs(a(n)),n,.5,21),
      explicit(-abs(a(n)),n,.5,21),
      border=true,color=black,fill_color=light_red,ra,
      label(["a(n)=(-1)^{(n+1)}/n",10,.9]),
      point_type=6,color=blue,points(abs(pa)),point_type=7,
      color=blue,points(pa))$
```



04. Functions

The function $y = f(x)$, $x \in D(f)$, i.e. $f\colon D(f) \to H(f)$.

- $D(f) \subset R$ A function of a real variable.
- $H(f) \subset R$ A real function.

Explicit form: • $y = f(x)$, $x \in D(f)$ (Analytical formula).

Parametric form: • $f\colon x = \varphi(t)$, $y = \psi(t)$, $t \in J$, $J \subset R$ (Auxiliary functions φ, ψ).

Implicit form: • $f\colon F(x, y) = 0$, conditions for $[x; y]$ (Implicit equation).

The function $f\colon y = |x|$, $x \in R$.

We can define the function $f\colon y = |x|$, $x \in R$, for example:

Explicit form: • $y = \sqrt{x^2}$, resp. • $y = \max\{-x, x\}$.

Parametric form: • $x = t$, $y = |t|$, $t \in R$, resp. • $x = t$, $y = \sqrt{t^2}$, $t \in R$.

Implicit form: • $y^2 - x^2 = 0$, $y \geq 0$, resp. • $y - |x| = 0$.

04. Functions

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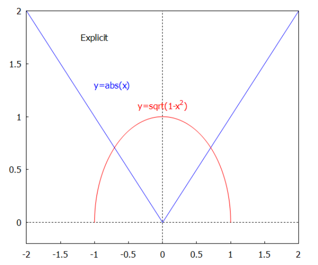
Implicit form: • $y^2 - x^2 = 0$, $y \geq 0$, resp. • $y - |x| = 0$.

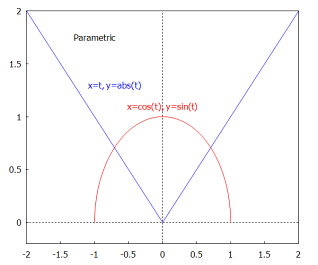
04. Functions

(%i1)
```
draw2d(xaxis=true,yaxis=true,
xrange=[-2,2],yrange=[-.2,2],
color=blue,explicit(abs(x),x,-2,2),
label(["y=abs(x)",-.75,1.3]),
color=red,explicit(sqrt(1-x^2),x,-1,1),
label(["y=sqrt(1-x^2)",0,1.1]),
color=black,label(["Explicit",-1,1.75]))$
```



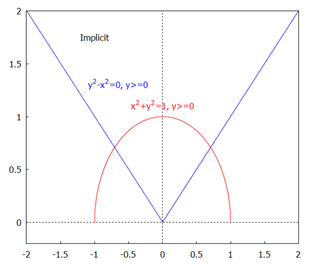
04. Functions

(%i1)
```
draw2d(xaxis=true,yaxis=true,
xrange=[-2,2],yrange=[-.2,2],
color=blue,parametric(t,abs(t),t,-2,2),
label(["x=t, y=abs(t)",-.7,1.3]),
color=red,nticks=100,parametric(cos(t),sin(t),t,0,%pi),
label(["x=cos(t), y=sin(t)",0,1.1]),
color=black,label(["Parametric",-1,1.75]))$
```



04. Functions

(%i1)
```
draw2d(xaxis=true,yaxis=true,
xrange=[-2,2],yrange=[-.2,2],
color=blue,implicit(y^2-x^2,x,-2,2,y,0,2),
label(["y^2-x^2=0, y>=0",-.65,1.3]),
color=red,implicit(x^2+y^2-1,x,-1,1,y,0,1),
label(["x^2+y^2=1, y>=0",0,1.1]),
color=black,label(["Implicit",-1,1.75]))$
```



05. Elementary Functions I

Elementary function is called each function created using the operations of addition, subtraction, multiplication, division or using composition of functions from basic elementary functions:

- $y = 1$,
- $y = x$,
- $y = e^x$,
- $y = \ln x$,
- $y = \sin x$,
- $y = \arcsin x$,
- $y = \arctan x$.

A polynomial of degree n is called

$$f_n\colon\ y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, \text{ where } a_0, a_1, \ldots, a_n \in R,\ n \in N \cup \{0\},\ a_n \neq 0.$$

- $f_0\colon\ y = a_0,\ a_0 \neq 0$ is called a **constant function**.
- $f_1\colon\ y = a_0 + a_1x,\ a_1 \neq 0$ is called a **linear function**.
- $f_2\colon\ y = a_0 + a_1x + a_2x^2,\ a_2 \neq 0$ is called a **quadratic function**.

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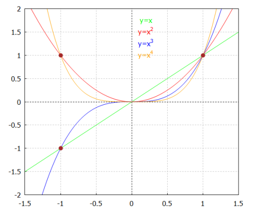
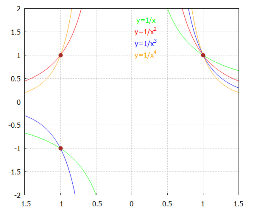
05. Elementary Functions I

(%i1)
```
draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[-1.5,1.5],yrange=[-2,2],
color=green,explicit(x,x,-1.5,1.5),
label(["y=x",.2,1.75]),
color=red,explicit(x^2,x,-1.5,1.5),
label(["y=x^2",.2,1.5]),
color=blue,explicit(x^3,x,-1.5,1.5),
label(["y=x^3",.2,1.25]),
color=orange,explicit(x^4,x,-1.5,1.5),
label(["y=x^4",.2,1]),color=brown,
point_type=7,points([[-1,-1],[-1,1],[1,1]]))$
```



05. Elementary Functions I

(%i1)
```
draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[-1.5,1.5],yrange=[-2,2],
color=green,explicit(1/x,x,-1.5,1.5),
label(["y=1/x",.2,1.75]),
color=red,explicit(1/x^2,x,-1.5,1.5),
label(["y=1/x^2",.2,1.5]),
color=blue,explicit(1/x^3,x,-1.5,1.5),
label(["y=1/x^3",.2,1.25]),
color=orange,explicit(1/x^4,x,-1.5,1.5),
label(["y=1/x^4",.2,1]),color=brown,
point_type=7,points([[-1,-1],[-1,1],[1,1]]))$
```



05. Elementary Functions I

Rational fractional function is called

$f\colon\ y = \frac{f_n(x)}{f_m(x)} = \frac{a_0+a_1x+a_2x^2+\cdots+a_nx^n}{b_0+b_1x+b_2x^2+\cdots+b_mx^m}$, where f_n, f_m are polynomials of degrees $n, m \in N \cup \{0\}$.

Power function is called

$f\colon\ y = x^r$, where $r \in R$, $r \neq 0$.

Exponential function with base $a > 0$ is called

$f\colon\ y = a^x$, $x \in R$.

- The most important one is $f\colon\ y = \exp x = \mathrm{e}^x$ with base e (Euler's number).
- The graph is called the **exponential curve** and passes through the points $[0; 1]$ and $[1; a]$.
- The graphs of the functions $y = a^x$, $y = a^{-x}$ are symmetric along the y axis.

05. Elementary Functions I

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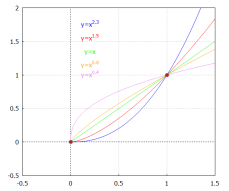
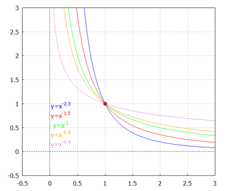
05. Elementary Functions I

(%i1)
```
draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[-.5,1.5],yrange=[-.5,2],
color=blue,explicit(x^2.3,x,0,1.5),
label(["y=x^{2.3}",.2,1.75]),
color=red,explicit(x^1.5,x,0,1.5),
label(["y=x^{1.5}",.2,1.55]),
color=green,explicit(x,x,-0,1.5),
label(["y=x",.2,1.35]),
color=orange,explicit(x^.8,x,0,1.5),
label(["y=x^{0.8}",.2,1.15]),
color=violet,explicit(x^.4,x,0,1.5),
label(["y=x^{0.4}",.2,1]),
color=brown,point_type=7,points([[0,0],[1,1]]))$
```



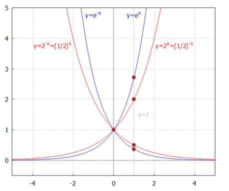
05. Elementary Functions I

```
(%i1) draw2d(grid=true,xaxis=true,yaxis=true,
      xrange=[-.5,3],yrange=[-.5,3],
      color=blue,explicit(x^-2.3,x,0,3),
      label(["y=x^{-2.3}",.2,.95]),
      color=red,explicit(x^-1.5,x,0,3),
      label(["y=x^{-1.5}",.2,.75]),
      color=green,explicit(x^-1,x,-0,3),
      label(["y=x^{-1}",.2,.55]),
      color=orange,explicit(x^-.8,x,0,3),
      label(["y=x^{-0.8}",.2,.35]),
      color=violet,explicit(x^-.4,x,0,3),
      label(["y=x^{-0.4}",.2,.15]),
      color=brown,point_type=7,points([[1,1]]))$
```



05. Elementary Functions I

(%i1)
```
draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[-5,5],yrange=[-.5,5],
color=blue,explicit(%e^x,x,-5,5),
label(["y=e^x",1,4.75]),
explicit(%e^(-x),x,-5,5),
label(["y=e^{-x}",-1,4.75]),
color=red,explicit(2^x,x,-5,5),
label(["y=2^x=(1/2)^{-x}",3,3.75]),
explicit(2^(-x),x,-5,5),
label(["y=2^{-x}=(1/2)^x",-3,3.75]),
color=grey,parametric(1,t,t,-.5,5),
label(["x=1",1.5,1.5]),color=brown,point_type=7,
points([[0,1],[1,1/2],[1,2],[1,1/%e],[1,%e]]))$
```



05. Elementary Functions I

```
(%i1) exp(x)+%e^x;exp(1);
```

(%o1) $2\%\, e^{x}$

(%o2) $\%\, e$

```
(%i5) log(x);log(2);log(%e);
```

(%o3) $\log(x)$

(%o4) $\log(2)$

(%o5) 1

```
(%i8) log_2(x):=log(x)/log(2);log_2(2);log_2(%e);
```

(%o6) $\log\_2(x) := \frac{\log(x)}{\log(2)}$

(%o7) 1

(%o8) $\frac{1}{\log(2)}$

05. Elementary Functions I

Logarithmic function with base $a > 0$, $a \neq 1$ is called

$$f\colon\ y = \log_a x,\ x \in (0; \infty).$$

- Logarithmic function $y = \log_a x$, $x \in (0; \infty)$ is the inverse of the exponential function $y = a^x$, $x \in R$ with the same base $a > 0$, $a \neq 1$ ($y = \log_a x \Leftrightarrow x = a^y$).
- For $a > 0$, $a \neq 1$ holds: $x = a^{\log_a x}$ for $x > 0$.
  $x = \log_a a^x$ for $x \in R$.
- The graph is called a logarithmic curve and passes through the points $[1; 0]$ and $[a; 1]$.
- The graphs of the functions $y = \log_a x$ and $y = \log_{a^{-1}} x$ are symmetric along the x axis.
- $a = 10$. $\Rightarrow$ Decadal logarithm, label $\log x = \log_{10} x$.
- $a = e$. $\Rightarrow$ Natural logarithm, label $\ln x = \log_e x$.
  `exp(x)=%e^x` and `log(x)` (natural logarithm) have the base e.
- If we want to calculate logarithm with another base, e.g. $\log_2 x$, we have to use construction $\log_2 x = \ln x / \ln 2$.

05. Elementary Functions I

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- For $a > 0$, $a \neq 1$ holds: $x = a^{\log_a x}$ for $x > 0$.
  $x = \log_a a^x$ for $x \in R$.
- The graph is called a logarithmic curve and passes through the points $[1; 0]$ and $[a; 1]$.
- The graphs of the functions $y = \log_a x$ and $y = \log_{a^{-1}} x$ are symmetric along the x axis.
- $a = 10$. $\Rightarrow$ Decadal logarithm, label $\log x = \log_{10} x$.
- $a = \mathrm{e}$. $\Rightarrow$ Natural logarithm, label $\ln x = \log_{\mathrm{e}} x$.
  `exp(x)=%e^x` and `log(x)` (natural logarithm) have the base e.
- If we want to calculate logarithm with another base, e.g. $\log_2 x$, we have to use construction $\log_2 x = \ln x / \ln 2$.

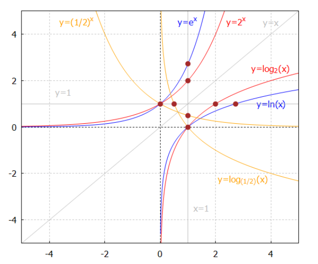
05. Elementary Functions I

(%i1)
```
draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[-5,5],yrange=[-5,5],
color=blue,explicit(exp(x),x,-5,5),
label(["y=e^x",1,4.5]),
explicit(log(x),x,.01,5),label(["y=ln(x)",4,1]),
color=red,explicit(2^x,x,-5,5),
label(["y=2^x",2.75,4.5]),
explicit(log(x)/log(2),x,.01,5),
label(["y=log_2(x)",4,2.5]),
color=orange,explicit((1/2)^x,x,-5,5),
label(["y=(1/2)^x",-3,4.5]),
explicit(-log(x)/log(2),x,.01,5),
label(["y=log_{(1/2)}(x)",3,-2.25]),
color=grey,parametric(t,t,t,-5,5),label(["y=x",4,4.5]),
parametric(1,t,t,-5,5),label(["x=1",1.5,-3.5]),
parametric(t,1,t,-5,5),label(["y=1",-3.5,1.5]),
color=brown,point_type=7,points([[1,0],[0,1],
[1,2],[2,1],[1,1/2],[1/2,1],[1,%e],[%e,1]]))$
```

05. Elementary Functions I

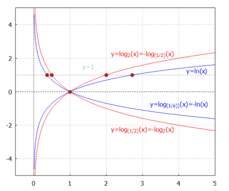
- Graphs of exponential and logarithmic functions from the previous page.

(%i1)
```
draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[-5,5],yrange=[-5,5],
color=blue,explicit(exp(x),x,-5,5),
label(["y=e^x",1,4.5]),
explicit(log(x),x,.01,5),label(["y=ln(x)",4,1]),
color=red,explicit(2^x,x,-5,5),
label(["y=2^x",2.75,...
```
the command continues (previous page)



05. Elementary Functions I

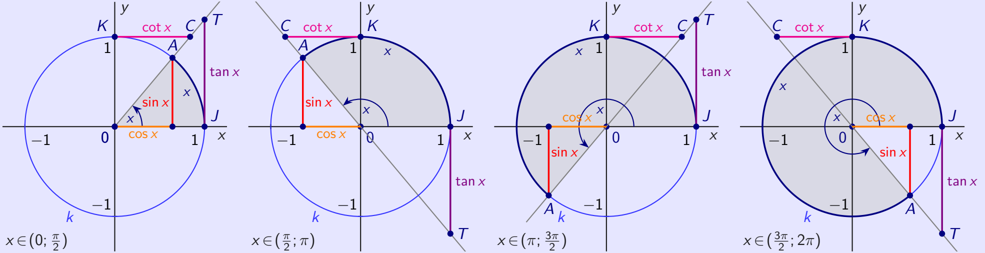
```
(%i1) draw2d(grid=true,xaxis=true,yaxis=true,
      xrange=[-.5,5],yrange=[-5,5],
      color=blue,explicit(log(x),x,.01,5),
      label(["y=ln(x)",4.5,1.25]),
      explicit(-log(x),x,.01,5),
      label(["y=log_{(1/e))}(x)=-ln(x)",4,-.75]),
      color=red,explicit(log(x)/log(2),x,.01,5),
      label(["y=log_2(x)=-log_{(1/2)}(x)",3,2.25]),
      explicit(-log(x)/log(2),x,.01,5),
      label(["y=log_{(1/2)}(x)=-log_2(x)",3,-2.25]),
      color=grey,parametric(t,1,t,-.5,5),
      label(["y=1",1.5,1.5]),color=brown,point_type=7,
      points([[1,0],[1/2,1],[2,1],[1/%e,1],[%e,1]]))$
```



06. Elementary Functions II

Trigonometric (goniometric) functions are:

- **Sine** $y = \sin x = |AA_x|$: $R \to \langle -1; 1 \rangle$.
- **Cosine** $y = \cos x = |OA_x|$: $R \to \langle -1; 1 \rangle$.
- **Tangent** $y = \tan x = \frac{\sin x}{\cos x} = |TJ|$: $R - \left\{ \frac{\pi}{2} + k\pi, k \in Z \right\} \to R$.
- **Cotangent** $y = \cot x = \frac{\cos x}{\sin x} = |CK|$: $R - \{ k\pi, k \in Z \} \to R$.



- The number π is called **Ludolf's**. Its value is approximately $3,141\,592\,654$.
- A circle with a radius $r = 1$ has a circumference of 2π.

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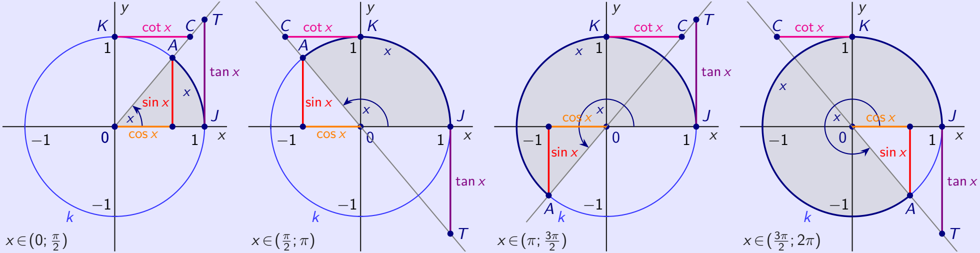
- **Cotangent** $y = \cot x = \frac{\cos x}{\sin x} = |CK|$: $R - \{k\pi, k \in Z\} \to R$.



- The number π is called **Ludolf's**. Its value is approximately $3,141\,592\,654$.
- A circle with a radius $r = 1$ has a circumference of 2π.

06. Elementary Functions II

- In Maxima, trigonometric functions have the form `sin(x)`, `cos(x)`, `tan(x)`, `cot(x)`.
- Arguments of trigonometric functions must be entered in radians.
- If we want to use degrees, we must first convert to radians.

```
(%i3) tangrad(x):=tan(x/180*%pi); tangrad(22.5);
      ratsimp(tangrad(22.5));
```

(%o1) $\mathrm{tangrad}(x) := \tan\left(\frac{x}{180}\pi\right)$

(%o2) $\tan(0.125\pi)$

```
rat: replaced 0.125 by 1/8 = 0.125
```

(%o3) $\tan\left(\frac{\pi}{8}\right)$

- To simplify work with trigonometric functions, we can use commands `trigsimp`, `trigrat`, `trigexpand`, `trigreduce` and packages `atrig1`, `ntrig` or `spangl`, which contain additional support for working with trigonometric functions.
- We have to load the packages into the system using the command `load`.

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(%i3) tangrad(x):=tan(x/180*%pi); tangrad(22.5);
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```

(%o1) $\mathrm{tangrad}(x) := \tan\left(\frac{x}{180}\pi\right)$

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rat: replaced 0.125 by 1/8 = 0.125
```

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06. Elementary Functions II

(%i1) `tan(%pi/4);tan(%pi/6);tan(%pi/8);`

(%o1) 1

(%o2) $\frac{1}{\sqrt{3}}$

(%o3) $\tan\left(\frac{\pi}{8}\right)$

(%i4) `ratsimp(tan(%pi/8));`

(%o4) $\tan\left(\frac{\pi}{8}\right)$

(%i5) `trigsimp(tan(%pi/8));`

(%o5) $\frac{\sin\left(\frac{\pi}{8}\right)}{\cos\left(\frac{\pi}{8}\right)}$

(%i6) `load(spangl);`

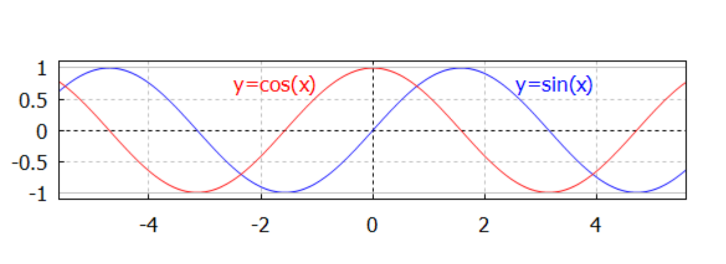
(%o6) *../share/trigonometry/spangl.mac*

(%i7) `tan(%pi/8);`

(%o7) $\sqrt{2}-1$

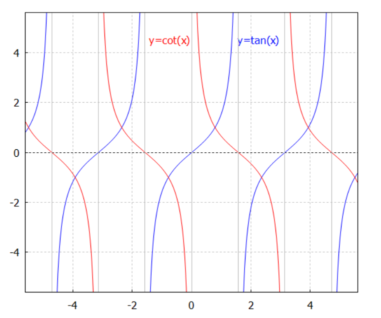
06. Elementary Functions II

(%i1)
```
draw2d(proportional_axes=xy,grid=true,
xaxis=true,yaxis=true,
xrange=[-1.75*%pi-.1,1.75*%pi+.1],yrange=[-1.1,1.1],
color=blue,explicit(sin(x),x,-3*%pi,3*%pi),
label(["y=sin(x)",3.25,.75]),
color=red,explicit(cos(x),x,-3*%pi,3*%pi),
label(["y=cos(x)",-1.75,.75]),
color=grey,parametric(t,1,t,-3*%pi,3*%pi),
parametric(t,-1,t,-3*%pi,3*%pi))$
```



06. Elementary Functions II

(%i1)
```
draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[-1.75*%pi-.1,1.75*%pi+.1],
yrange=[-1.75*%pi-.1,1.75*%pi+.1],
color=blue,explicit(tan(x),x,-3*%pi,3*%pi),
label(["y=tan(x)",2.25,4.5]),
color=red,explicit(cot(x),x,-3*%pi,3*%pi),
label(["y=cot(x)",-.75,4.5]),
color=grey,     /* asymptotes */
parametric(0,t,t,-6,6),
parametric(%pi/2,t,t,-6,6),
parametric(-%pi/2,t,t,-6,6),
parametric(%pi,t,t,-6,6),
parametric(-%pi,t,t,-6,6),
parametric(3*%pi/2,t,t,-6,6),
parametric(-3*%pi/2,t,t,-6,6))$
```



06. Elementary Functions II

Sum formulas for sine and cosine. $x, y \in R$.

- $\sin(x \pm y) = \sin x \cdot \cos y \pm \cos x \cdot \sin y$.
- $\cos(x \pm y) = \cos x \cdot \cos y \mp \sin x \cdot \sin y$.
- $\sin 2x = \sin(x + x) = 2 \sin x \cdot \cos x$.
- $\cos 2x = \cos(x + x) = \cos^2 x - \sin^2 x$.
- $\sin^2 x = \frac{1 - \cos 2x}{2}$.
- $\cos^2 x = \frac{1 + \cos 2x}{2}$.
- $\sin^2 x + \cos^2 x = 1$.

Cyclometric functions are inverses of trigonometric functions:

- **Arcsine** $y = \arcsin x$: $\langle -1; 1 \rangle \to \langle \frac{\pi}{2}; \frac{\pi}{2} \rangle$.
- **Arccosine** $y = \arccos x$: $\langle -1; 1 \rangle \to \langle 0; \pi \rangle$.
- **Arctangent** $y = \arctan x$: $R \to \left(-\frac{\pi}{2}; \frac{\pi}{2} \right)$.
- **Arccotangent** $y = \operatorname{arccot} x$: $R \to (0; \pi)$.

- There are no inverse functions for trigonometric functions because they are not injective. It is necessary to narrow them appropriately.

06. Elementary Functions II

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- $\sin^2 x + \cos^2 x = 1$.

Cyclometric functions are inverses of trigonometric functions:

- **Arcsine** $y = \arcsin x$: $\langle -1; 1\rangle \to \left\langle \frac{\pi}{2}; \frac{\pi}{2} \right\rangle$.
- **Arccosine** $y = \arccos x$: $\langle -1; 1\rangle \to \langle 0; \pi\rangle$.
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- There are no inverse functions for trigonometric functions because they are not injective. It is necessary to narrow them appropriately.

06. Elementary Functions II

- Cyclometric functions have the form Maxima `asin(x)`, `acos(x)`, `atan(x)`, `acot(x)`.
- At this point we can mention the function `atan2(x,y)` defined by the relation $\arctan \frac{x}{y}$.

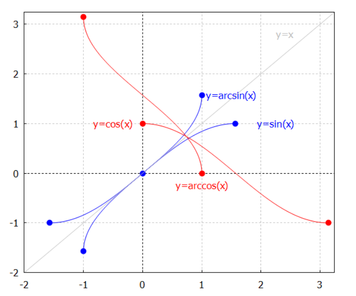
```
(%i4) asin(1);asin(1),numer;
      acos(1);acos(1),numer;
(%o1) π/2
(%o2) 1.570796326794897
(%o1) 0
(%o2) 0.0
(%i7) atan2(2,4);atan(1/2);atan(1/2),numer;
(%o5) atan(1/2)
(%o6) atan(1/2)
(%o7) 0.463647609000806
```

Sum formulas for cyclometric functions.

- $\arcsin x + \arccos x = \frac{\pi}{2}$ for $x \in \langle -1; 1 \rangle$.
- $\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$ for $x \in R$.

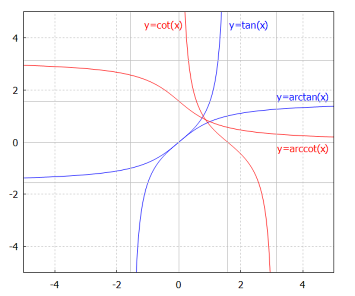
06. Elementary Functions II

```
(%i1) draw2d(grid=true,xaxis=true,yaxis=true,
      xrange=[-2,%pi+.1],yrange=[-2,%pi+.1],
      color=blue,explicit(sin(x),x,-%pi/2,%pi/2),
      label(["y=sin(x)",2.25,1]),explicit(asin(x),x,-1,1),
      label(["y=arcsin(x)",1.5,%pi/2]),
      point_type=7, points([[0,0],[1,%pi/2],
      [-1,-%pi/2],[%pi/2,1],[-%pi/2,-1]]),
      color=red,explicit(cos(x),x,0,%pi),
      label(["y=cos(x)",-.5,1]),
      explicit(acos(x),x,-1,1),
      label(["y=arccos(x)",1,-.25]),
      point_type=7,
      points([[0,1],[1,0],
      [%pi,-1],[-1,%pi]]),
      color=grey,
      parametric(t,t,t,-5,5),
      label(["y=x",2.4,2.8]))$
```



06. Elementary Functions II

(%i1)
```
draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[-5,5],yrange=[-5,5],
color=blue,explicit(tan(x),x,-%pi/2+.01,%pi/2-.01),
label(["y=tan(x)",2.25,4.5]),explicit(atan(x),x,-5,5),
label(["y=arctan(x)",4,1.75]),color=grey,
parametric(t,-%pi/2,t,-5,5),parametric(t,%pi/2,t,-5,5),
parametric(-%pi/2,t,t,-5,5),parametric(%pi/2,t,t,-5,5),
color=red,explicit(cot(x),x,.01,%pi-.01),
label(["y=cot(x)",-.5,4.5]),
explicit(%pi/2-atan(x),x,-5,5),
label(["y=arccot(x)",4,-.25]),
color=grey,
parametric(t,0,t,-5,5),
parametric(t,%pi,t,-5,5),
parametric(0,t,t,-5,5),
parametric(%pi,t,t,-5,5))$
```



06. Elementary Functions II

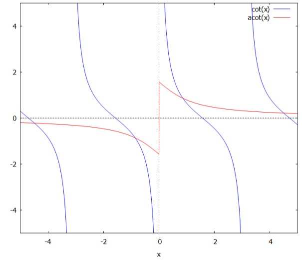
- Beware of the inaccurate interpretation of the Arccotangent function in the wxMaxima.

```
(%i4) acot(-1)$
      print("acot(-1)=",acot(-1),"is bad")$
      %pi/2-atan(-1)$
      print("acot(-1)=",%pi/2-atan(-1),"is ok")$
```

$acot(-1) = -\frac{\pi}{4}$ *is bad*

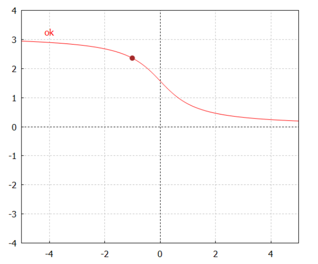
$acot(-1) = -\frac{3\pi}{4}$ *is ok*

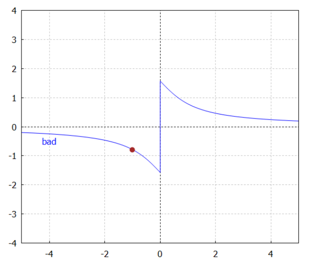
```
(%i5) plot2d([cot(x),acot(x)],[x,-5,5],[y,-5,5])$
```



06. Elementary Functions II

```
(%i1) draw2d(grid=true,xaxis=true,yaxis=true,
      xrange=[-5,5],yrange=[-4,4],color=red,
      explicit(%pi/2-atan(x),x,-5,5),label(["ok",-4,3.25]),
      color=brown,point_type=7,
      points([[-1,%pi/2-atan(-1)]]))$
(%i2) draw2d(grid=true,xaxis=true,yaxis=true,
      xrange=[-5,5],yrange=[-4,4],color=blue,
      explicit(acot(x),x,-5,5),label(["bad",-4,-.5]),
      color=brown,point_type=7,points([[-1,acot(-1)]]))$
```





06. Elementary Functions II

Hyperbolic functions are:

- **Hyperbolic sine** $y = \sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2\,e^x}$: $R \to R$.
- **Hyperbolic cosine** $y = \cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2\,e^x}$: $R \to \langle 1; \infty)$.
- **Hyperbolic tangent** $y = \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$: $R \to (-1; 1)$.
- **Hyperbolic cotangent** $y = \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$: $R - \{0\} \to R - \langle -1; 1\rangle$.

- Hyperbolic functions have similar properties to trigonometric functions.

Sum formulas for hyperbolic sine and hyperbolic cosine. $x, y \in R$.

- $\sinh(x \pm y) = \sinh x \cdot \cosh y \pm \cosh x \cdot \sinh y$.
- $\cosh(x \pm y) = \cosh x \cdot \cosh y \pm \sinh x \cdot \sinh y$.
- $\sinh x \pm \cosh x = \frac{e^x - e^{-x}}{2} \pm \frac{e^x + e^{-x}}{2} = \pm e^{\pm x}$.
- $\sinh^2 x = \frac{\cosh 2x - 1}{2}$.
- $\cosh^2 x = \frac{\cosh 2x + 1}{2}$.
- $\cosh^2 x - \sinh^2 x = 1$.

06. Elementary Functions II

Hyperbolic functions are:

- **Hyperbolic sine** $y = \sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2\,e^x}\colon \quad R \to R.$
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- $\cosh^2 x = \frac{\cosh 2x + 1}{2}.$
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06. Elementary Functions II

Moivre formula. $x \in R,\ n \in N.$

- $(\cosh x \pm \sinh x)^n = \cosh nx \pm \sinh nx$

- Hyperbolic functions are `sinh(x)`, `cosh(x)`, `tanh(x)`, `coth(x)` and to them the inverse hyperbolometric functions are `asinh(x)`, `acosh(x)`, `atanh(x)`, `acoth(x)`.

```
(%i4) sinh(x);cosh(0);tanh(0);coth(1),numer;
(%o1) sinh(x)
(%o2) 1
(%o3) 0
(%o4) 1.313035285499331
(%i8) asinh(x);acosh(1);atanh(0);acoth(1.3),numer;
(%o5) asinh(x)
(%o6) 0
(%o7) 0
(%o8) 1.01844096363052
```

06. Elementary Functions II

Hyperbolometric (Inverse hyperbolic) functions are inverses of hyperbolic functions:

- **Inverse hyperbolic sine**

$$y = \operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right):\quad R \to R.$$

- **Inverse hyperbolic cosine**

$$y = \operatorname{arcosh} x = \ln\left(x + \sqrt{x^2-1}\right):\quad \langle 1;\infty) \to \langle 0;\infty).$$

- **Inverse hyperbolic tangent**

$$y = \operatorname{artanh} x = \tfrac{1}{2}\ln\tfrac{1+x}{1-x}:\qquad (-1;1) \to R.$$

- **Inverse hyperbolic cotangent**

$$y = \operatorname{arcoth} x = \tfrac{1}{2}\ln\tfrac{x+1}{x-1}:\qquad R - \langle -1;1\rangle \to R - \{0\}.$$

```
(%i3) ash(x):=log(x+sqrt(x^2+1))$
      a:2$ asinh(a)-ash(a),numer;
(%o3) 0.0
```

07. Limit of a function

- When investigating a function, it is necessary to characterize its local properties at different intervals and around different important points.
- The function f does not have to be defined at the point around which we investigate it.

Point $a \in R^*$ is an **accumulation point** of the set $A \subset R$,
if for every neighborhood $O(a)$ there exists $x \in O(a) \cap A$, $x \neq a$.

The following definition of limits using sequences is called Heine's.

The function f **has a limit** $b \in R^*$ at the point $a \in R^*$, label $\lim\limits_{x \to a} f(x) = b$, if:

- a is the accumulation point of the set $D(f)$.
- For all $\{x_n\}_{n=1}^{\infty} \subset D(f)$, $x_n \neq a$, $\{x_n\}_{n=1}^{\infty} \longrightarrow a$ holds $\{f(x_n)\}_{n=1}^{\infty} \longrightarrow b$.

If $\lim\limits_{x \to a} f(x) = b$, then there exists (at least one) $\{x_n\}_{n=1}^{\infty} \longrightarrow a$, $x_n \in D(f) - \{a\}$
and $\lim\limits_{x \to a} f(x) = \lim\limits_{n \to \infty} f(x_n)$ holds.

07. Limit of a function

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and $\lim\limits_{x \to a} f(x) = \lim\limits_{n \to \infty} f(x_n)$ holds.

07. Limit of a function

$a \in R^*$ is the accumulation point od sets $D(f)$ and $D(g)$, $O(a)$ is the neighborhood.

$\forall x \in O(a),\ x \neq a$:
- $f(x) = g(x)$. $\Rightarrow$ • $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x)$ if they exist.
- $f(x) \leq g(x)$. $\Rightarrow$ • $\lim\limits_{x\to a} f(x) \leq \lim\limits_{x\to a} g(x)$ if they exist.

$\forall x \in O(a),\ x \neq a$:
- $f(x) < g(x)$. $\Rightarrow$ • $\lim\limits_{x\to a} f(x) \leq \lim\limits_{x\to a} g(x)$ if they exist.

Two Policemen and a Drunk Theorem.

$a \in R^*$ is the accumulation point od sets $D(f)$, $D(g)$ and $D(h)$, $O(a)$ is the neighborhood.

- $\forall x \in O(a),\ x \neq a$: $h(x) \leq f(x) \leq g(x)$.
- $\lim\limits_{x\to a} h(x) = \lim\limits_{x\to a} g(x) = b$, where $b \in R^*$.

$\Rightarrow$ • There exists $\lim\limits_{x\to a} f(x) = b$.

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0.$$

- ∞ is the accumulation point of the domain $D(f) = R - \{0\}$ of the function $f\colon\ y = \frac{\sin x}{x}$.
- $x > 0. \Rightarrow -\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}. \Rightarrow 0 = -\lim\limits_{x\to\infty} \frac{1}{x} \leq \lim\limits_{x\to\infty} \frac{\sin x}{x} \leq \lim\limits_{x\to\infty} \frac{1}{x} = 0. \Rightarrow \lim\limits_{x\to\infty} \frac{\sin x}{x} = 0.$

07. Limit of a function

- $\lim_{x\to\infty} \frac{\sin x}{x} = 0.$

```
(%i1) limit(sin(x)/x,x,inf);
(%o1) 0
```

```
(%i1) f(x):=sin(x)/x$ for i:1 thru 10 do
          (x:100^i, print(x," ",ev(f(x),numer)))$
```

$100 \quad -0.0050636564110975$

$10000 \quad -3.0561438888825 \cdot 10^{-5}$

$1000000 \quad -3.499935021712929 \cdot 10^{-7}$

$100000000 \quad 9.31639027109726 \cdot 10^{-9}$

$10000000000 \quad -4.875060250875107 \cdot 10^{-11}$

$1000000000000 \quad -6.112387023768895 \cdot 10^{-13}$

$100000000000000 \quad -2.094083074964523 \cdot 10^{-15}$

$10000000000000000 \quad 7.796880066069787 \cdot 10^{-17}$

$1000000000000000000 \quad -9.929693207404051 \cdot 10^{-19}$

$100000000000000000000 \quad -6.452512852657808 \cdot 10^{-21}$

07. Limit of a function

$\lim\limits_{x\to 0} \frac{\sin x}{x} = 1.$

- 0 is the accumulation point of the domain $D(f) = R - \{0\}$ of the function $f\colon\ y = \frac{\sin x}{x}$.
- $0 < x < \frac{\pi}{2}$. $\quad\Rightarrow 0 < \sin x < x < \tan x. \quad \Rightarrow 1 = \frac{\sin x}{\sin x} < \frac{x}{\sin x} < \frac{\frac{\sin x}{\cos x}}{\sin x} = \frac{1}{\cos x}.$
- $-\frac{\pi}{2} < x < 0$. $\quad\Rightarrow \tan x < x < \sin x < 0. \quad \Rightarrow \frac{1}{\cos x} = \frac{\frac{\sin x}{\cos x}}{\sin x} > \frac{x}{\sin x} > \frac{\sin x}{\sin x} = 1.$
- $x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right) - \{0\}$. $\quad\Rightarrow 1 < \frac{x}{\sin x} < \frac{1}{\cos x}.$

$$\Rightarrow 1 = \lim_{x\to 0} 1 \le \lim_{x\to 0} \frac{x}{\sin x} \le \lim_{x\to 0} \frac{1}{\cos x} = 1. \Rightarrow \lim_{x\to 0} \frac{x}{\sin x} = 1.$$

```
(%i1) limit(sin(x)/x,x,0);
(%o1) 1
(%i2) limit(sin(x)/x,x,inf);
(%o2) 0
(%i3) limit(sin(x)/x,x,minf);
(%o3) 0
```

07. Limit of a function

- $\lim_{x\to\infty} x\left(\sqrt[x]{e}-1\right) = 1.$

```
(%i2) limit(x*(%e^(1/x)-1),x,0);limit(x*(%e^(1/x)-1),x,inf);
(%o1) und                /* undefined */
(%o2) 1
```

- $\lim_{x\to 2} \frac{x-2}{x^2-3x+2} = \lim_{x\to 2} \frac{x-2}{(x-2)(x-1)} = \lim_{x\to 2} \frac{1}{x-1} = \frac{1}{2-1} = 1.$
- $\lim_{x\to 0} \frac{3x+2x^{-1}}{x+4x^{-1}} = \lim_{x\to 0} \frac{3x+2x^{-1}}{x+4x^{-1}} \cdot \frac{x}{x} = \lim_{x\to 0} \frac{3x^2+2}{x^2+4} = \frac{3\cdot 0+2}{0+4} = \frac{1}{2}.$
- $\lim_{x\to 2} \frac{x^2-3x+2}{x^2-2x} = \lim_{x\to 2} \frac{(x-1)(x-2)}{x(x-2)} = \lim_{x\to 2} \frac{x-1}{x} = \frac{2-1}{2} = \frac{1}{2}.$

```
(%i3) limit((x-2)/(x^2-3*x+2),x,2);
      limit((3*x+2*1/x)/(x+4*1/x),x,0);
      limit((x^2-3*x+2)/(x^2-2*x),x,2);
(%o1) 1
(%o2) 1/2
(%o3) 1/2
```

07. Limit of a function

- $\lim_{x\to\infty} x\left(\sqrt[x]{e}-1\right) = 1.$

```
(%i2)  limit(x*(%e^(1/x)-1),x,0);limit(x*(%e^(1/x)-1),x,inf);
(%o1)  und                /* undefined */
(%o2)  1
```

- $\lim_{x\to 2} \frac{x-2}{x^2-3x+2} = \lim_{x\to 2} \frac{x-2}{(x-2)(x-1)} = \lim_{x\to 2} \frac{1}{x-1} = \frac{1}{2-1} = 1.$
- $\lim_{x\to 0} \frac{3x+2x^{-1}}{x+4x^{-1}} = \lim_{x\to 0} \frac{3x+2x^{-1}}{x+4x^{-1}} \cdot \frac{x}{x} = \lim_{x\to 0} \frac{3x^2+2}{x^2+4} = \frac{3\cdot 0+2}{0+4} = \frac{1}{2}.$
- $\lim_{x\to 2} \frac{x^2-3x+2}{x^2-2x} = \lim_{x\to 2} \frac{(x-1)(x-2)}{x(x-2)} = \lim_{x\to 2} \frac{x-1}{x} = \frac{2-1}{2} = \frac{1}{2}.$

```
(%i3)  limit((x-2)/(x^2-3*x+2),x,2);
       limit((3*x+2*1/x)/(x+4*1/x),x,0);
       limit((x^2-3*x+2)/(x^2-2*x),x,2);
(%o1)  1
(%o2)  1/2
(%o3)  1/2
```

07. Limit of a function

- $$\lim_{x\to 0}\frac{x}{\sqrt{1+x}-\sqrt{1-x}}=\lim_{x\to 0}\frac{x\cdot(\sqrt{1+x}+\sqrt{1-x})}{(\sqrt{1+x}-\sqrt{1-x})\cdot(\sqrt{1+x}+\sqrt{1-x})}=\lim_{x\to 0}\frac{x\cdot(\sqrt{1+x}+\sqrt{1-x})}{(1+x)-(1-x)}$$
$$=\lim_{x\to 0}\frac{x\cdot(\sqrt{1+x}+\sqrt{1-x})}{2x}=\lim_{x\to 0}\frac{\sqrt{1+x}+\sqrt{1-x}}{2}=\frac{\sqrt{1+0}+\sqrt{1-0}}{2}=\frac{1+1}{2}=1.$$

- $$\lim_{x\to 0}\frac{1-\sqrt{1-x}}{x}=\lim_{x\to 0}\frac{(1-\sqrt{1-x})\cdot(1+\sqrt{1-x})}{x\cdot(1+\sqrt{1-x})}=\lim_{x\to 0}\frac{1-(1-x)}{x+\sqrt{x^2-x^3}}=\lim_{x\to 0}\frac{x}{x+\sqrt{x^2-x^3}}$$
$$=\lim_{x\to 0}\frac{x}{x+x\cdot\sqrt{1-x}}=\lim_{x\to 0}\frac{1}{1+\sqrt{1-x}}=\frac{1}{1+\sqrt{1-0}}=\frac{1}{2}.$$

- $$\lim_{x\to\infty}\frac{\sqrt{x^2-1}+\sqrt{x^2+1}}{x}=\lim_{x\to\infty}\left(\sqrt{\frac{x^2-1}{x^2}}+\sqrt{\frac{x^2+1}{x^2}}\right)=\lim_{x\to\infty}\left(\sqrt{1-\frac{1}{x^2}}+\sqrt{1+\frac{1}{x^2}}\right)$$
$$=\sqrt{1-\frac{1}{\infty}}+\sqrt{1+\frac{1}{\infty}}=\sqrt{1-0}+\sqrt{1+0}=1+1=2.$$

```
(%i3)  limit(x/(sqrt(1+x)-sqrt(1-x)),x,0);
       limit((1-sqrt(1-x))/x,x,0);
       limit((\sqrt(x^2-1)+sqrt(x^2+1))/x,x,inf);
(%o1)  1
(%o2)  1/2
(%o3)  2
```

07. Limit of a function

- $\lim\limits_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1} = \begin{bmatrix} \text{Subst.} & x = z^{12} \\ x \to 1, & z \to 1 \end{bmatrix} = \lim\limits_{z\to 1} \frac{\sqrt[3]{z^{12}}-1}{\sqrt[4]{z^{12}}-1} = \lim\limits_{z\to 1} \frac{z^4-1}{z^3-1}$

$$= \lim_{z\to 1} \frac{(z-1)(z^3+z^2+z+1)}{(z-1)(z^2+z+1)} = \lim_{z\to 1} \frac{z^3+z^2+z+1}{z^2+z+1} = \frac{1+1+1+1}{1+1+1} = \frac{4}{3}.$$

```
(%i1) limit((x^(1/3)-1)/(x^(1/4)-1),x,1);
(%o1) 4/3
```

If we use the substitution $x = z^{12}$, we can simplify the first limit.

```
(%i2) f(x):=(x^(1/3)-1)/(x^(1/4)-1)$
      g(z):=subst(z^12,x,f(x))$
      'limit(g(z),z,1); limit(g(z),z,1);
(%o1) lim_{z->1} (z^4-1)/(z^2·|z|-1)      /* z is positive, z=|z| */
(%o2) 4/3
```

07. Limit of a function

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```

07. Limit of a function

- $\lim\limits_{x\to\infty}\left(\frac{5x^2}{x^2-1}+2^{\frac{1}{x}}\right)=\lim\limits_{x\to\infty}\frac{5}{1-x^{-2}}+\lim\limits_{x\to\infty}2^{\frac{1}{x}}=\frac{5}{1-\infty^{-2}}+2^0=\frac{5}{1-0}+1=6.$

- $\lim\limits_{x\to 0^+}x^{\frac{a}{\ln x}}=\lim\limits_{x\to 0^+}e^{\ln x^{\frac{a}{\ln x}}}=\lim\limits_{x\to 0^+}e^{\frac{a}{\ln x}\cdot\ln x}=\lim\limits_{x\to 0^+}e^a=e^a$ for $a\in R$.

```
(%i4) limit(5*x^2/(x^2-1)+2^(1/x),x,inf);
      limit(x^(a/log(x)),x,0,plus);
      limit(x^(2/log(x)),x,0,plus);
      limit(x^(-2/log(x)),x,0,plus);
```

(%o1) 6
(%o2) e^a
(%o3) e^2
(%o4) e^{-2}

In the last example, we calculated the limit of the expression 0^0 (so called indefinite expression).

Among the **indefinite expressions** (we count them using limits) are:

- $\infty-\infty$, • $\pm\infty\cdot 0$, • $\frac{0}{0}$, • $\frac{1}{0}$, • $\frac{\pm\infty}{0}$, • $\frac{\pm\infty}{\pm\infty}$, • 0^0, • $0^{\pm\infty}$, • $1^{\pm\infty}$, • $(\pm\infty)^0$.

07. Limit of a function

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07. Limit of a function

- $\lim\limits_{x\to\infty} x\big(\ln(x+2)-\ln x\big) = \lim\limits_{x\to\infty} x\cdot\ln\frac{x+2}{x} = \lim\limits_{x\to\infty}\ln\left(1+\frac{2}{x}\right)^x = \ln e^2 = 2.$

- $\lim\limits_{x\to 0}\frac{x}{\ln(1+tx)} = \begin{bmatrix}\text{Subst.} & z=tx\\ x\to 0, & z\to 0\end{bmatrix} = \lim\limits_{z\to 0}\frac{\frac{z}{t}}{\ln(1+z)} = \frac{1}{t}\cdot\lim\limits_{z\to 0}\frac{1}{\frac{1}{z}\cdot\ln(1+z)}$
  $= \frac{1}{t}\cdot\lim\limits_{z\to 0}\frac{1}{\ln(1+z)^{\frac{1}{z}}} = \frac{1}{t}\cdot\frac{1}{\ln e} = \frac{1}{t}\cdot\frac{1}{1} = \frac{1}{t}$ for $t\in R,\ t\neq 0.$

- $\lim\limits_{x\to\infty}\left(\frac{3x-2}{3x+1}\right)^x = \lim\limits_{x\to\infty}\left(\frac{3x+1-3}{3x+1}\right)^{\frac{3x+1-1}{3}} = \begin{bmatrix}\text{Subst. } z=3x+1\\ x\to\infty, \quad z\to\infty\end{bmatrix} = \lim\limits_{z\to\infty}\left(\frac{z-3}{z}\right)^{\frac{z-1}{3}}$
  $= \lim\limits_{z\to\infty}\left[\left(1-\frac{3}{z}\right)^z\right]^{\frac{z-1}{3z}} = \left[e^{-3}\right]^{\frac{1}{3}} = e^{-1} = \frac{1}{e}.$

```
(%i3) limit(x*(log(x+2)-log(x)),x,inf);
      limit(x/log(1+t*x),x,0);
      limit(((3*x-2)/(3*x+1))^x,x,inf);
```

(%o1) 2
(%o2) $\frac{1}{t}$
(%o3) e^{-1}

07. Limit of a function

When investigating the function f , it is important to examine its properties at non-eigenpoints:

- For $x \to \pm\infty$.
- In the neighborhood $O(a)$ of the point $a \in R$,
  for which holds $\lim\limits_{x\to a^-} f(x) = \pm\infty$ or $\lim\limits_{x\to a^+} f(x) = \pm\infty$.

A function $y = f(x)$, $x \in D(f)$, a point $a \in R$.

- The line $x = a$ is called **asymptote without slope** (**vertical**) of the graph f,
  if $\lim\limits_{x\to a^-} f(x) = \pm\infty$ or $\lim\limits_{x\to a^+} f(x) = \pm\infty$ (at least one of the limits is infinite).
- The line $y = kx + q$ is called the **asymptote with slope** of the graph f,
  if $\lim\limits_{x\to -\infty} [f(x) - (kx+q)] = 0$ or $\lim\limits_{x\to \infty} [f(x) - (kx+q)] = 0$.

Specially the asymptote $y = q$ is called **horizontal asymptote**,
i.e. $k = 0$ and $\lim\limits_{x\to -\infty} f(x) = q$ or $\lim\limits_{x\to \infty} f(x) = q$.

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07. Limit of a function

The functions $y = f(x)$, $x \in D(f)$ and a domain $D(f)$ is an unbounded set.

- The line $y = kx + q$ is an asymptote with the slope of the graph f.

$$\Leftrightarrow \quad \bullet \text{ Exist } \lim_{x\to\pm\infty} \frac{f(x)}{x} = k,\ \lim_{x\to\pm\infty} [f(x) - kx] = q,\ k, q \in R.$$

$$\lim_{x\to\infty} \frac{f(x)-(kx+q)}{x} = \lim_{x\to\infty} \left[\frac{f(x)}{x} - k - \frac{q}{x}\right] = 0. \qquad \Rightarrow \lim_{x\to\infty} \frac{f(x)}{x} = k.$$

$$\lim_{x\to\infty} [f(x) - (kx+q)] = \lim_{x\to\infty} [(f(x) - kx) - q] = 0. \qquad \Rightarrow \lim_{x\to\infty} [f(x) - kx] = q.$$

A function $f(x) = \frac{2x^2+x+1}{8x}$, $x \in R$.

- $k = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{2x^2+x+1}{8x^2} = \lim_{x\to\pm\infty} \frac{x^2(2+\frac{1}{x}+\frac{1}{x^2})}{8x^2} = \lim_{x\to\pm\infty} \frac{2+\frac{1}{x}+\frac{1}{x^2}}{8} = \frac{2+0+0}{8} = \frac{1}{4}.$
- $q = \lim_{x\to\pm\infty} [f(x) - kx] = \lim_{x\to\pm\infty} \left[\frac{2x^2+x+1}{8x} - \frac{x}{4}\right]$

$$= \lim_{x\to\pm\infty} \left[\frac{2x^2+x+1}{8x} - \frac{2x^2}{8x}\right] = \lim_{x\to\pm\infty} \frac{x+1}{8x} = \lim_{x\to\pm\infty} \left[\frac{1}{8} + \frac{1}{8x}\right] = \frac{1}{8}.$$

- The line $y = \frac{x}{4} + \frac{1}{8}$ is an asymptote with the slope $\frac{1}{4}$.

07. Limit of a function

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- $q = \lim\limits_{x\to\pm\infty} [f(x) - kx] = \lim\limits_{x\to\pm\infty} \left[\frac{2x^2+x+1}{8x} - \frac{x}{4}\right]$

$$= \lim_{x\to\pm\infty} \left[\frac{2x^2+x+1}{8x} - \frac{2x^2}{8x}\right] = \lim_{x\to\pm\infty} \frac{x+1}{8x} = \lim_{x\to\pm\infty} \left[\frac{1}{8} + \frac{1}{8x}\right] = \frac{1}{8}.$$

- The line $y = \frac{x}{4} + \frac{1}{8}$ is an asymptote with the slope $\frac{1}{4}$.

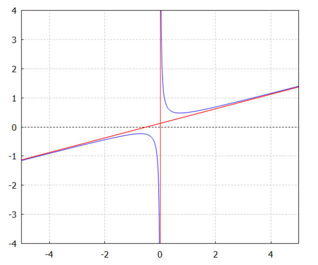
07. Limit of a function

```
(%i10) f(x):=(2*x^2+x+1)/(8*x); km:limit(f(x)/x,x,minf)$
       kp:limit(f(x)/x,x,inf)$
       qm:limit(f(x)-km*x,x,minf)$ qp:limit(f(x)-kp*x,x,inf)$
       dm(x):=km*x+qm$ dp(x):=kp*x+qp$ dm(x);dp(x);
       draw2d(grid=true,xaxis=true,yaxis=true,
       xrange=[-5,5],yrange=[-4,4],
       color=blue,explicit(f(x),x,-8,0),
       explicit(f(x),x,0,8),
       color=red,parametric(0,t,t,-5,5),
       explicit(dm(x),x,-8,8),
       explicit(dp(x),x,-8,8))$
```

(%o1) $f(x) := \frac{2x^2+x+1}{8x}$

(%o8) $\frac{x}{4} + \frac{1}{8}$

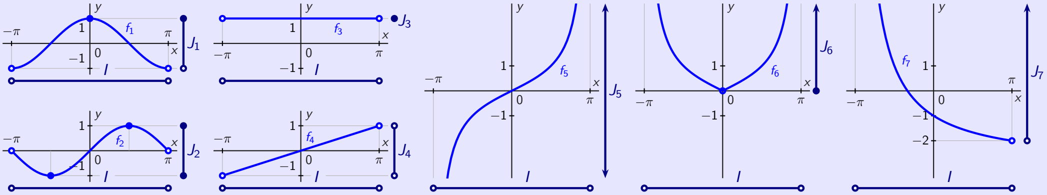
(%o9) $\frac{x}{4} + \frac{1}{8}$



08. Continuity of function

If A function f is continuous on the interval $I \subset R$, then the set $f(I)$ is an interval.

- $I = \langle a; b \rangle$ is a closed interval. $\Rightarrow$ • The set $f(I)$ is a closed interval.
- I is not a closed interval. $\Rightarrow$ • The set $f(I)$ can be an interval of different types.



- $f_1(x) = \cos x$: $(-\pi; \pi) \to J_1 = (-1; 1\rangle$.
- $f_2(x) = \sin x$: $(-\pi; \pi) \to J_2 = \langle -1; 1\rangle$.
- $f_3(x) = 1$: $(-\pi; \pi) \to J_3 = \{1\}$.
- $f_4(x) = \frac{x}{\pi}$: $(-\pi; \pi) \to J_4 = (-1; 1)$.
- $f_5(x) = \tan \frac{x}{2}$: $(-\pi; \pi) \to J_5 = (-\infty; \infty)$.
- $f_6(x) = \left|\tan \frac{x}{2}\right|$: $(-\pi; \pi) \to J_6 = \langle 0; \infty)$.
- $f_7(x) = -\frac{3x+\pi}{x+\pi}$: $(-\pi; \pi) \to J_7 = (-2; \infty)$.

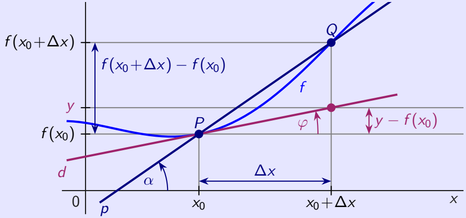
03. Differential calculus

Mathematical Analysis supported by wxMaxima

01. Derivative of a real function

The function $y = f(x)$, $x \in D(f)$ is continuous.

- The points $P = [x_0; f(x_0)]$, $Q = [x_0 + \Delta x; f(x_0 + \Delta x)]$ lie on graph f.
- The line PQ has the slope $\tan\alpha = \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$.
- The tangent line to f at point P has the form d_P: $y - f(x_0) = \tan\varphi \cdot \Delta x$,
  where $\tan\varphi = \frac{y - f(x_0)}{\Delta x}$ is its slope.



- $Q \to P$. $\Rightarrow$
- $\Delta x \to 0$.
  - $x_0 + \Delta x \to x_0$,
  - $f(x_0 + \Delta x) \to f(x_0)$.
  - $\alpha \to \varphi$,
  - $\tan\alpha \to \tan\varphi$.
  - $PQ \to d_P$ (the line PQ to the tangent line).

- The tangent line has the slope $\tan\varphi = \lim_{\alpha\to\varphi} \tan\alpha = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$.

Geometric meaning of the derivative of a function at a point. – The slope of the tangent line.

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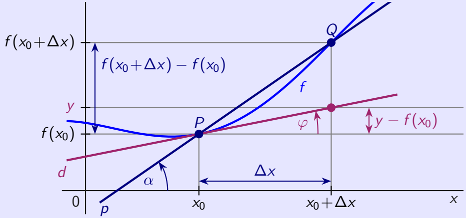

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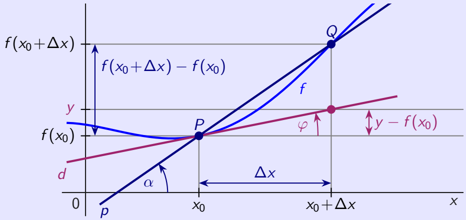

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Geometric meaning of the derivative of a function at a point. – The slope of the tangent line.

01. Derivative of a real function

A function $y = f(x)$, $x \in D(f)$ has a **derivative at the point** $x_0 \in D(f)$,

label $f'(x_0)$, resp. $y'(x_0)$ or $f'(x_0) = \frac{\mathrm{d}f(x_0)}{\mathrm{d}x}$, resp. $y'(x_0) = \frac{\mathrm{d}y(x_0)}{\mathrm{d}x}$ using differentials,

if it exists • $\lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \begin{bmatrix} \text{Subst. } h = x - x_0 \\ x \to x_0, \quad h \to 0 \end{bmatrix} = \lim\limits_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0).$

- $f'(x_0) \in R$. Eigen (finite)
- $f'(x_0) = \infty$ or $f'(x_0) = -\infty$. Non-eigen (infinite)

} derivative of the f at the point x_0.

A function $y = f(x)$, $x \in D(f)$, point $x_o \in D(f)$.

- There exists $f'(x_0) \in R$ (finite). $\Rightarrow$ • f is continuous at the point x_0.

The continuity of the function f at the point x_0 does not guarantee the existence of $f'(x_0)$.

The function $f\colon\ y = |x|$ is continuous at the point $x_0 = 0$.

- But, there does not exist $f'(0) = \lim\limits_{x \to 0} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x \to 0} \frac{|x|-|0|}{x-0} = \lim\limits_{x \to 0} \frac{|x|}{x} = \begin{cases} \lim\limits_{x \to 0^+} \frac{x}{x} = 1. \\ \lim\limits_{x \to 0^+} \frac{-x}{x} = -1. \end{cases}$

01. Derivative of a real function

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01. Derivative of a real function

$f'(x_0)$ represents geometrically the slope of the tangent line to the graph f at the point x_0.

- $f'(x_0) \in R$. Tangent line d: $y = f(x_0) + f'(x_0)(x - x_0)$ with slope $f'(x_0)$.
- $f'(x_0) = \pm\infty$ and f is continuous at the point x_0.
  Tangent line d: $x = x_0$ without slope (vertical).

We calculate the derivative of the function $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$.

(%i1) `f(x):=log(x+sqrt(x^2+1));`

(%o1) $f(x) := \log\left(x + \sqrt{x^2+1}\right)$

(%i3) `f_1(x):=diff(f(x),x);f_1(x);`

(%o2) $f\_1(x) := \frac{d}{dx} f(x)$

(%o3) $\frac{\frac{x}{\sqrt{x^2+1}}+1}{\sqrt{x^2+1}+x}$

(%i4) `ratsimp(f1(x));`

(%o4) $\frac{\sqrt{x^2+1}+x}{x\sqrt{x^2+1}+x^2+1}$

01. Derivative of a real function

$f'(x_0)$ represents geometrically the slope of the tangent line to the graph f at the point x_0.

- $f'(x_0) \in R$. Tangent line d: $y = f(x_0) + f'(x_0)(x - x_0)$ with slope $f'(x_0)$.
- $f'(x_0) = \pm\infty$ and f is continuous at the point x_0.
  Tangent line d: $x = x_0$ without slope (vertical).

We calculate the derivative of the function $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$.

(%i1) `f(x):=log(x+sqrt(x^2+1));`

(%o1) $f(x) := \log\left(x + \sqrt{x^2+1}\right)$

(%i3) `f_1(x):=diff(f(x),x);f_1(x);`

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(%o3) $\frac{\frac{x}{\sqrt{x^2+1}}+1}{\sqrt{x^2+1}+x}$

(%i4) `ratsimp(f1(x));`

(%o4) $\frac{\sqrt{x^2+1}+x}{x\sqrt{x^2+1}+x^2+1}$

01. Derivative of a real function

We have calculated the derivative of $f'(x)$ (in the previous example), but we have not succeeded in simplifying it appropriately. We will use the command `subst`.

```
(%i1) f(x):=log(x+sqrt(x^2+1));
```

(%o1) $f(x) := \log\left(x + \sqrt{x^2+1}\right)$

```
(%i3) f_1(x):=diff(f(x),x);f_1(x);
```

(%o2) $f\_1(x) := \frac{d}{dx} f(x)$

(%o3) $\frac{\frac{x}{\sqrt{x^2+1}}+1}{\sqrt{x^2+1}+x}$

```
(%i4) fp:subst(a,sqrt(x^2+1),f1(x));
```

(fp) $\frac{\frac{x}{a}+1}{x+a}$

```
(%i5) ratsimp(fp);
```

(%o5) $\frac{1}{a}$

```
(%i6) subst(sqrt(x^2+1),a,ratsimp(fp));
```

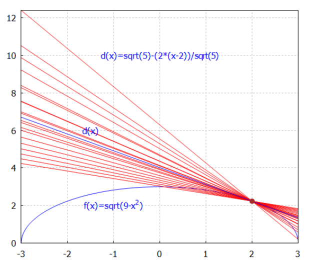
(%o6) $\frac{1}{\sqrt{x^2+1}}$

01. Derivative of a real function

We determine the tangent line to the semicircle $y = \sqrt{9 - x^2}$ at the point 2.

(%i8)
```
f(x):=sqrt(9-x^2)$ p(a,b):=(f(b)-f(a))/(b-a)$
s(x,a,b):=p(a,b)*(x-a)+f(a)$
S:makelist(explicit(s(x,2,-.15+.25*i),x,-3,3),i,1,20)$
f1(x):=diff(f(x),x,1)$
d(x):=f(2)+subst(2,x,f1(x))*(x-2)$
print("Secant d(x)=",d(x)," in point 2 is a blue")$
draw2d(grid=true,xaxis=true,
color=blue,explicit(f(x),x,-3,3),color=red,S,
color=blue,explicit(d(x),x,-3,3),
point_type=7,color=brown,
points([[2,f(2)]]),
color=blue,label(["d(x)",-1.5,6]),
label(["f(x)=sqrt(9-x^2)",-1,2]),
label([concat("d(x)=",
     string(d(x))),0,10]))$
```

Secant $d(x) = \sqrt{5} - \frac{2(x-2)}{\sqrt{5}}$ in point 2 is a blue

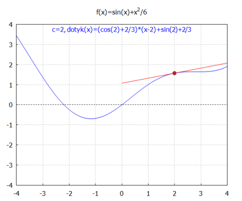


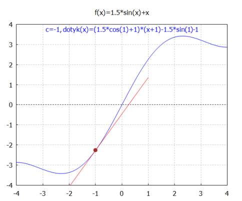
01. Derivative of a real function

The following example constructs a tangent line to the graph of the function f at point c.

```
(%i6) c:2$ f(x):=x^2/6+sin(x)$ f1(x):=diff(f(x),x,1)$
      d(x):=f(c)+subst(c,x,f1(x))*(x-c)$
      print("Secant y=d(x)=",d(x)," in point",c)$
      draw2d(grid=true,xaxis=true,xrange=[-4,4],
      yrange=[-4,4],color=blue,explicit(f(x),x,-4,4),
      color=red,explicit(d(x),x,c-2,c+2),
      point_type=7,color=brown,points([[c,f(c)]]),
      color=blue,title=concat("f(x)=",string(f(x))),
      label([concat("c=",string(c),",
          d(x)=",string(d(x))),0,3.75]))$
```

Secant $y = d(x) = (\cos(2) + \frac{2}{3})(x-2) + \sin(2) + \frac{2}{3}$ in point 2





02. Differential of a function and derivatives of higher orders

The best local linear approximation.

A function $y = f(x)$, $x \in D(f)$ is differentiable at the point $x_0 \in D(f)$.

- Approximation of the function f in the neighborhood $O(x_0)$ using at the point x_0 the tangent line d: $y = f(x_0) + f'(x_0)(x - x_0) = f(x_0)$, $x \in O(x_0)$ is the best of all approximations using a linear function (straight line).

Calculate approximately $\sqrt[6]{1.06}$.

- Let us denote $f(x) = \sqrt[6]{x} = x^{1/6}$, $x > 0$, $x_0 = 1$.
- $f(x_0) = f(1) = 1$.
- $f'(x) = [x^{1/6}]' = \frac{1}{6}x^{-5/6} = \frac{1}{6\sqrt[6]{x^5}}$, $x > 0$.
- $f'(x_0) = f'(1) = \frac{1}{6}$.
- Let $O(1)$ be such that $1.06 \in O(1)$.

$\Rightarrow$ • $\sqrt[6]{x} = f(x) \approx f(1) + f'(1) \cdot (x - 1) = 1 + \frac{x-1}{6} = \frac{6+x-1}{6} = \frac{x+5}{6}$.

$\Rightarrow$ • $\sqrt[6]{1.06} = f(1.06) \approx \frac{1.06+5}{6} = \frac{6.06}{6} = 1.01$.

Exactly $\sqrt[6]{1.06} = 1.0097588$, calculation error < 0.00025.

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02. Differential of a function and derivatives of higher orders

```
(%i9)  c:1.06$ f(x):=x^(1/6)$ s:1$ f1(x):=diff(f(x),x,1)$
       p(x):=f(s)+subst(s,x,f1(x))*(x-s)$
       h(c):=print("c=",c,'f(c),"=",float(f(c)),"approx",
       subst(c,x,float(p(x))))$ fpprintprec:8$ p(x); h(c)$
```

(%o8) $\frac{x-1}{6}+1$

$c = 1.06 \quad f(1.06) = 1.0097588$ approx 1.01

The variable `fpprintprec:8` sets the output to 8 digits.
The approximation of the function f makes sense only for x near the point x_0.

```
(%i18) h(.9)$h(1.1)$h(1.2)$h(1.5)$h(2.0)$h(4)$h(9)$h(30)$h(64)$
```

$c = 0.9 \quad f(0.9) = 0.98259319$ approx 0.98333333
$c = 1.1 \quad f(1.1) = 1.0160119$ approx 1.0166667
$c = 1.2 \quad f(1.2) = 1.0308533$ approx 1.0333333
$c = 1.5 \quad f(1.5) = 1.0699132$ approx 1.0833333
$c = 2.0 \quad f(2.0) = 1.122462$ approx 1.1666667
$c = 4 \quad f(4) = 1.259921$ approx 1.5
$c = 9 \quad f(9) = 1.4422496$ approx 2.3333333
$c = 30 \quad f(30) = 1.7627344$ approx 5.8333333
$c = 64 \quad f(64) = 2.0$ approx 11.5

02. Differential of a function and derivatives of higher orders

```
(%i9) c:1.06$ f(x):=x^(1/6)$ s:1$ f1(x):=diff(f(x),x,1)$
      p(x):=f(s)+subst(s,x,f1(x))*(x-s)$
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      subst(c,x,float(p(x))))$ fpprintprec:8$ p(x); h(c)$
```

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02. Differential of a function and derivatives of higher orders

Calculate approximately $\sqrt[6]{1.06}$ (other solution).

- Let us denote $f(x) = \sqrt[6]{x+1} = (x+1)^{1/6}$, $x > -1$, $x_0 = 0$. $\Rightarrow$ • $f(x_0) = f(0) = 1$.
- $f'(x) = [(x+1)^{1/6}]' = \frac{1}{6}(x+1)^{-5/6} = \frac{1}{6\sqrt[6]{(x+1)^5}}$, $x > -1$. $\Rightarrow$ • $f'(x_0) = f'(0) = \frac{1}{6}$.
- Let $O(0)$ be such that $0.06 \in O(0)$.

$$\Rightarrow \bullet \ \sqrt[6]{x+1} = f(x) \approx f(0) + f'(0) \cdot (x-0) = 1 + \frac{x}{6} = \frac{x+6}{6}.$$

$$\Rightarrow \bullet \ \sqrt[6]{1.06} = f(0.06) \approx \frac{0.06+6}{6} = \frac{6.06}{6} = 1.01.$$

We will change the first commands `c:.06`, `f(x):=(x+1)^(1/6)`, `s:0` in the previous example.

```
(%i9) c:.06$ f(x):=(x+1)^(1/6)$ s:0$ f1(x):=diff(f(x),x,1)$
      p(x):=f(s)+subst(s,x,f1(x))*(x-s)$
      h(c):=print("c=",c,'f(c),"=",float(f(c)),"approx",
      subst(c,x,float(p(x))))$ fpprintprec:8$ p(x); h(c)$
```

(%o8) $\frac{x}{6} + 1$

$c = 0.06$ $f(0.06) = 1.0097588$ approx 1.01

02. Differential of a function and derivatives of higher orders

Calculate approximately $\sqrt[6]{1.06}$ (other solution).

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(%i9) c:.06$ f(x):=(x+1)^(1/6)$ s:0$ f1(x):=diff(f(x),x,1)$
      p(x):=f(s)+subst(s,x,f1(x))*(x-s)$
      h(c):=print("c=",c,'f(c),"=",float(f(c)),"approx",
      subst(c,x,float(p(x))))$ fpprintprec:8$ p(x); h(c)$
```

(%o8) $\frac{x}{6} + 1$

$c = 0.06 \quad f(0.06) = 1.0097588$ approx 1.01

02. Differential of a function and derivatives of higher orders

Calculating $f^{(n)}$, $n \in N$ can be very laborious in general.

A function $y = x^k$, $x \in R$, where $k \in N$.

- $[x^k]^{(n)} = k(k-1)\cdots(k-n+1)x^{k-n}$, $x \in R$ for $n = 1, 2, \ldots, k$,
  $[x^k]' = kx^{k-1}$, $[x^k]'' = k(k-1)x^{k-2}$, $[x^k]''' = k(k-1)(k-2)x^{k-3}$, $\ldots$, $[x^k]^{(k)} = k!$.
- $[x^k]^{(n)} = 0$, $x \in R$ for $n = k+1, k+2, k+3, \ldots$,
  $[x^k]^{(k+1)} = [k!]' = 0$, $[x^k]^{(k+2)} = [x^k]^{(k+3)} = [0]' = 0$, $\ldots$

```
(%i9) f(x,k):=x^k;fn(x,k,n):=diff(f(x,k),x,n)$
      fn(x,k,1);fn(x,k,2);fn(x,k,k);
      fn(x,5,1);fn(x,5,2);fn(x,5,5);fn(x,5,6);
```

(%o1) $f(x,k) := x^k$

(%o3) kx^{k-1}

(%o4) $(k-1)kx^{k-2}$

(%o5) $\frac{d^k}{dx^k}x^k$

(%o6) $5x^4$

(%o7) $20x^3$

(%o8) 120

(%o9) 0

03. Applications of the derivative of a function

$\lim\limits_{x\to 2} \frac{x^3-8}{x-2} = 12.$

- $f(x) = x^3 - 8,\ x\in R,\ g(x) = x - 2,\ x\in R.$ $O(2)$ can be chosen arbitrarily, e.g. $O(2) = R$.
- $f'(x) = 3x^2,\ g'(x) = 1$ for $x\in R - \{2\}$.
- $\lim\limits_{x\to 2} \frac{f'(x)}{g'(x)} = \lim\limits_{x\to 2} \frac{3x^2}{1} = 12.$
- $\lim\limits_{x\to 2} (x^3 - 8) = \lim\limits_{x\to 2} (x - 2) = 0.$

$\Rightarrow$

- $\lim\limits_{x\to 2} \frac{x^3-8}{x-2} = \lim\limits_{x\to 2} \frac{3x^2}{1} = 12.$

```
(%i9) f(x):=(x^3-8)/(x-2)$
      fc(x):=num(f(x))$ fc(x);fm(x):=denom(f(x))$ fm(x);
      'limit(f(x),x,2); limit(f(x),x,2);
      'limit(diff(fc(x),x,1)/diff(fm(x),x,1),x,2);
      limit(diff(fc(x),x,1)/diff(fm(x),x,1),x,2);
```

(%o3) $x^3 - 8$

(%o5) $x - 2$

(%o6) $\lim\limits_{x\to 2} \frac{x^3-8}{x-2}$

(%o7) 12

(%o8) $3 \lim\limits_{x\to 2} x^2$

(%o9) 12

03. Applications of the derivative of a function

- $\lim\limits_{x\to\infty} \frac{\ln x}{x} = [L'H\frac{\infty}{\infty}] = \lim\limits_{x\to\infty} \frac{[\ln x]'}{[x]'} = \lim\limits_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim\limits_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = 0.$

```
(%i4) f(x):=log(x)/x$
      fc(x):=num(f(x))$
      fm(x):=denom(f(x))$
      limit(diff(fc(x),x,1)/diff(fm(x),x,1),x,inf);
(%o4) 0
```

- $\lim\limits_{x\to 0} \frac{x-\sin x}{x^3} = [L'H\frac{0}{0}] = \lim\limits_{x\to 0} \frac{1-\cos x}{3x^2}$

  $= [L'H\frac{0}{0}] = \lim\limits_{x\to 0} \frac{0-(-\sin x)}{6x} = \lim\limits_{x\to 0} \frac{\sin x}{6x}$

  $= [L'H\frac{0}{0}] = \lim\limits_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}.$

- $\lim\limits_{x\to\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}} = [L'H\frac{\infty}{\infty}] = \lim\limits_{x\to\infty} \frac{e^x-e^{-x}}{e^x+e^{-x}}$

  $= [L'H\frac{\infty}{\infty}] = \lim\limits_{x\to\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}} = \cdots$

We cannot use L'Hospital's rule.

03. Applications of the derivative of a function

A function $y = f(x)$, $x \in D(f)$, a point $x_0 \in D(f)$, $n \in N$,
a neighborhood $O(x_0) \subset D(f)$, $f'(x_0), f''(x_0), \ldots, f^{(n)}(x_0) \in R$ (finite).

Taylor polynomial of degree n of the function f centered at the point x_0 is defined as

- $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)\cdot(x-x_0)^k}{k!} = f(x_0) + \frac{f'(x_0)\cdot(x-x_0)^1}{1!} + \cdots + \frac{f^{(n)}(x_0)\cdot(x-x_0)^n}{n!}$, $x \in O(x_0)$.

For $x_0 = 0$ is called **Maclaurin polynomial**

- $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)\cdot x^k}{k!} = f(0) + \frac{f'(0)\cdot x}{1!} + \frac{f''(0)\cdot x^2}{2!} + \cdots + \frac{f^{(n)}(0)\cdot x^n}{n!}$, $x \in O(0)$.

Let us denote $h = x - x_0$, $x = x_0 + h$, $h \in O(0)$.

- $T_n(x_0 + h) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)\cdot h^k}{k!} = f(x_0) + \frac{f'(x_0)\cdot h}{1!} + \frac{f''(x_0)\cdot h^2}{2!} + \cdots + \frac{f^{(n)}(x_0)\cdot h^n}{n!}$, $h \in O(0)$.

The remainder $R_n(x) = f(x) - T_n(x)$ expresses the approximation error of f using $T_n(x)$.

- $R_n(x) = \frac{f^{(n+1)}(x_0+\theta(x-x_0))\cdot(x-x_0)^{n+1}}{(n+1)!}$, $x \in O(x_0)$, where $\theta \in (0;1)$. (Lagrangian form.)

03. Applications of the derivative of a function

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- $T_n(x)=\sum\limits_{k=0}^{n}\frac{f^{(k)}(x_0)\cdot(x-x_0)^k}{k!}=f(x_0)+\frac{f'(x_0)\cdot(x-x_0)^1}{1!}+\cdots+\frac{f^{(n)}(x_0)\cdot(x-x_0)^n}{n!}$, $x\in O(x_0)$.

For $x_0=0$ is called **Maclaurin polynomial**

- $T_n(x)=\sum\limits_{k=0}^{n}\frac{f^{(k)}(0)\cdot x^k}{k!}=f(0)+\frac{f'(0)\cdot x}{1!}+\frac{f''(0)\cdot x^2}{2!}+\cdots+\frac{f^{(n)}(0)\cdot x^n}{n!}$, $x\in O(0)$.

Let us denote $h=x-x_0$, $x=x_0+h$, $h\in O(0)$.

- $T_n(x_0+h)=\sum\limits_{k=0}^{n}\frac{f^{(k)}(x_0)\cdot h^k}{k!}=f(x_0)+\frac{f'(x_0)\cdot h}{1!}+\frac{f''(x_0)\cdot h^2}{2!}+\cdots+\frac{f^{(n)}(x_0)\cdot h^n}{n!}$, $h\in O(0)$.

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- $R_n(x)=\frac{f^{(n+1)}(x_0+\theta(x-x_0))\cdot(x-x_0)^{n+1}}{(n+1)!}$, $x\in O(x_0)$, where $\theta\in(0;1)$. (Lagrangian form.)

03. Applications of the derivative of a function

We calculate the Taylor polynomial $T_n(x)$ of the function $f(x) = \sqrt{x^2+1}$.

- Manual derivation is quite laborious.

```
(%i2) f(x):=sqrt(x^2+1)$ print("f(x)=", f(x),
      ", f'(x)=", diff(f(x),x),
      ", f''(x)=", ratsimp(diff(f(x),x,2)),
      ", f'''(x)=", ratsimp(diff(f(x),x,3)))$
```

$$f(x) = \sqrt{x^2+1},\ f'(x) = \frac{x}{\sqrt{x^2+1}},\ f''(x) = \frac{\sqrt{x^2+1}}{x^4+2x^2+1},\ f'''(x) = -\frac{3x\sqrt{x^2+1}}{x^6+3x^4+3x^2+1}$$

```
(%i3) taylor(f(x),x,0,1);
```

$$1+\cdots$$

```
(%i4) taylor(f(x),x,0,2);
```

$$1+\frac{x^2}{2}+\cdots$$

```
(%i5) taylor(f(x),x,0,3);
```

$$1+\frac{x^2}{2}+\cdots$$

```
(%i6) taylor(f(x),x,0,4);
```

$$1+\frac{x^2}{2}-\frac{x^4}{8}+\cdots$$

```
(%i7) taylor(f(x),x,0,18);
```

$$1+\frac{x^2}{2}-\frac{x^4}{8}+\frac{x^6}{16}-\frac{5x^8}{128}+\frac{7x^{10}}{256}-\frac{21x^{12}}{1024}+\frac{33x^{14}}{2048}-\frac{429x^{16}}{32768}+\frac{715x^{18}}{65536}+\cdots$$

03. Applications of the derivative of a function

We are calculating the $T_n(x)$ of the function $f(x) = \sqrt{x^2+1}$.

- The polynomial tp1 is of the ninth degree (practically eighth degree), therefore the output command coeff(tp1,x,10)) is the number 0.
- The polynomial tp2 is of the tenth degree and the command output coeff(tp2,x,10)) is real coefficient $c_{10} = 7/256$.

```
(%i1) f(x):=sqrt(x^2+1)$
(%i2) tp1:taylor(f(x),x,0,9);
```

(tp1) $1+\frac{x^2}{2}-\frac{x^4}{8}+\frac{x^6}{16}-\frac{5x^8}{128}+\cdots$

```
(%i3) print("c_3=",coeff(tp1,x,3),
        ", c_4=",coeff(tp1,x,4),
        ", c_10=",coeff(tp1,x,10))$
```

c_3= 0, c_4= $-\frac{1}{8}$, c_10= 0

```
(%i4) tp2:taylor(f(x),x,0,10);
```

(tp2) $1+\frac{x^2}{2}-\frac{x^4}{8}+\frac{x^6}{16}-\frac{5x^8}{128}+\frac{7x^{10}}{256}+\cdots$

```
(%i5) print("c_3=",coeff(tp2,x,3),
        ", c_4=",coeff(tp2,x,4),
        ", c_10=",coeff(tp2,x,10))$
```

c_3= 0, c_4= $-\frac{1}{8}$, c_10= $\frac{7}{256}$

03. Applications of the derivative of a function

$f(x) = \ln x$, $x \in (0; \infty)$, $x_0 = 1$, $f(1) = 0$, $n \in N$.

- $f^{(1)}(x) = \frac{1}{x} = x^{-1}$, $x > 0$.
- $f^{(2)}(x) = -x^{-2}$, $x > 0$.
- $f^{(3)}(x) = 2x^{-3}$, $x > 0$.
- $f^{(4)}(x) = -3 \cdot 2x^{-4}$, $x > 0$.

  ...
- $f^{(k)}(x) = (-1)^{k-1}(k-1)!x^{-k}$, $x > 0$, $k = 1, 2, 3, \ldots, n$.

- $f^{(1)}(1) = 1 = 0!$.
- $f^{(2)}(1) = -1 = -1!$.
- $f^{(3)}(1) = 2 \cdot 1 = 2!$.
- $f^{(4)}(1) = -3 \cdot 2 \cdot 1 = -3!$.
- $f^{(k)}(1) = (-1)^{k-1}(k-1)!$.

$\Rightarrow$ • $T_n(x) = 0 + \sum_{k=1}^{n} \frac{f^{(k)}(1) \cdot (x-1)^k}{k!} = \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)! \cdot (x-1)^k}{k!} = \sum_{k=1}^{n} \frac{(-1)^{k-1} \cdot (x-1)^k}{k}$, $x \in O(1)$.

```
(%i1) taylor(log(x),x,1,5);
```

(%o1) $x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} + \cdots$

```
(%i2) taylor(log(x),x,1,8);
```

(%o2) $x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6} + \frac{(x-1)^7}{7} - \frac{(x-1)^8}{8} + \cdots$

03. Applications of the derivative of a function

Sometimes it is more convenient to express $f(x) = \ln x$ in the form of a Maclaurin polynomial.

- $f(x) = \ln x,\ x \in (0;\infty),\ x_0 = 1.$
  $T_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}\cdot(x-1)^k}{k},\ x \in O(1),\ n \in N.$
- $x = t+1,\ f(x) = f(t+1) = \ln(t+1),\ t \in (-1;\infty).$
  $T_n(x) = T_n(t+1) = \sum_{k=1}^{n} \frac{(-1)^{k-1}\cdot(t+1-1)^k}{k} = \sum_{k=1}^{n} \frac{(-1)^{k-1}\cdot t^k}{k},\ x \in O(1),\ t \in O(0),\ n \in N.$
- Maclaurin polynomial of the function $f(x) = \ln(x+1)$ of degree $n \in N$:
  $T_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n-1}x^n}{n} = \sum_{k=1}^{n} \frac{(-1)^{k-1}x^k}{k},\ x \in O(0).$

(%i1) `taylor(log(x+1),x,0,8);`

(%o1) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \cdots$

(%i2) `taylor(log(x),x,1,8);`

(%o2) $x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6} + \frac{(x-1)^7}{7} - \frac{(x-1)^8}{8} + \cdots$

(%i3) `taylor(log(x+1),x,1,8);`

(%o3) $\log(2) + \frac{x-1}{2} - \frac{(x-1)^2}{8} + \frac{(x-1)^3}{24} - \frac{(x-1)^4}{64} + \frac{(x-1)^5}{160} - \frac{(x-1)^6}{384} + \frac{(x-1)^7}{896} - \frac{(x-1)^8}{2048} + \cdots$

03. Applications of the derivative of a function

We can approximate the functions $y = e^x$, $y = \sin x$, $y = \cos x$ using **Maclaurin polynomial** for all $x \in R$. We can achieve the necessary accuracy with a sufficiently large degree of n.

- $f(x) = e^x,\ x \in R.$
  $T_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} = \sum_{i=0}^{n} \frac{x^i}{i!},\ x \in R,\ n \in N.$
- $f(x) = \sin x,\ x \in R.$
  $T_{2k+1}(x) = 0 + \frac{x}{1!} + 0 + \frac{-x^3}{3!} + 0 + \frac{x^5}{5!} + \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sum_{i=0}^{k} \frac{(-1)^i x^{2i+1}}{(2i+1)!},\ x \in R,\ n \in N.$
- $f(x) = \cos x,\ x \in R$
  $T_{2k}(x) = 1 + 0 + \frac{-x^2}{2!} + 0 + \frac{x^4}{4!} + 0 + \cdots + \frac{(-1)^k x^{2k}}{(2k)!} = \sum_{i=0}^{k} \frac{(-1)^i x^{2i}}{(2i)!},\ x \in R,\ n \in N.$

```
(%i1) taylor(exp(x),x,0,10);
```
(%o1) $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880} + \frac{x^{10}}{3628800} + \cdots$

```
(%i2) taylor(sin(x),x,0,10);
```
(%o2) $x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} + \cdots$

```
(%i3) taylor(cos(x),x,0,10);
```
(%o3) $1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800} + \cdots$

03. Applications of the derivative of a function

We find the Maclaurin polynomial of degree $n \in N$ of the function $f(x) = e^{(x^2)}$, $x \in R$.

- If we denote $g(t) = e^t$, $t \in R$, $t = x^2$, then $f(x) = e^{(x^2)} = g(x^2) = g(t) = e^t$, $t \geq 0$.
- For the Maclaurin polynomial $P_n(t)$ of the function $g(t)$, $t \geq 0$ and the Maclaurin polynomial $T_{2n}(x)$ of the function $f(x)$, $x \in R$ holds:

$$P_n(t) = \sum_{i=0}^{n} \frac{t^i}{i!} = \sum_{i=0}^{n} \frac{(x^2)^i}{i!} = \sum_{i=0}^{n} \frac{x^{2i}}{i!} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots + \frac{x^{2n}}{n!} = T_{2n}(x).$$

The Maclaurin polynomial of degree $2n$ of the function $f(x) = e^{(x^2)}$, $x \in R$ has the form

- $T_{2n}(x) = \sum_{i=0}^{n} \frac{x^{2i}}{i!} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots + \frac{x^{2n}}{n!}$, $x \in R$.

```
(%i1) taylor(exp(x^2),x,0,10);
```

(%o1) $1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24} + \frac{x^{10}}{120} + \cdots$

```
(%i3) subst(x^2,t,taylor(exp(t),t,0,5));
      subst(x^2,t,taylor(exp(t),t,0,10));
```

(%o2) $\frac{x^{10}}{120} + \frac{x^8}{24} + \frac{x^6}{6} + \frac{x^4}{2} + x^2 + 1$

(%o3) $\frac{x^{20}}{3628800} + \frac{x^{18}}{362880} + \frac{x^{16}}{40320} + \frac{x^{14}}{5040} + \frac{x^{12}}{720} + \frac{x^{10}}{120} + \frac{x^8}{24} + \frac{x^6}{6} + \frac{x^4}{2} + x^2 + 1$

04. Investigation of behaviour of functions

An important part of the investigation of the behaviour of the function is the determination of the intervals, for which this function is monotonic.

A function f is continuous on an interval I, for all $x \in I$ there exists $f'(x) \in R$ (finite).

The function f is on I			For all $x \in I$ holds	
	• increasing.	$\Leftrightarrow$		• $f'(x) > 0$.
	• decreasing.	$\Leftrightarrow$		• $f'(x) < 0$.
	• non-decreasing.	$\Leftrightarrow$		• $f'(x) \geq 0$.
	• non-increasing.	$\Leftrightarrow$		• $f'(x) \leq 0$.
	• constant.	$\Leftrightarrow$		• $f'(x) = 0$.

A necessary condition for the existence of a local extremum.

A function $y = f(x)$, $x \in D(f)$, $x_0 \in D(f)$ is an interior point of $D(f)$, there exists $f'(x_0)$.

- The function f has a local extremum at the point x_0. $\Rightarrow$ • $f'(x_0) = 0$.

- The function f can have a local extremum at a point where the derivative does not exist.

- $f'(x_0) = 0$ does not guarantee a local extremum of the function f at the point $x_0 \in D(f)$.

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	• non-increasing.	$\Leftrightarrow$		• $f'(x) \leq 0$.
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04. Investigation of behaviour of functions

A function f is continuous on an interval I, for all $x \in I$ there exists $f'(x) \in R$ (finite).

f is on I	• convex.	$\Leftrightarrow$	f' is on I	• non-decreasing.
	• concave.	$\Leftrightarrow$		• non-increasing.
	• strictly convex.	$\Leftrightarrow$		• increasing.
	• strictly concave.	$\Leftrightarrow$		• descreasing.

A function f is continuous on an interval I, for all $x \in I$ there exists $f''(x) \in R$ (finite).

f is on I	• convex.	$\Leftrightarrow$	For all $x \in I$ holds	• $f''(x) > 0$.
	• concave.	$\Leftrightarrow$		• $f''(x) < 0$.
	• strictly convex.	$\Leftrightarrow$		• $f''(x) \geq 0$.
	• strictly concave.	$\Leftrightarrow$		• $f''(x) \leq 0$.

When investigating the convexity and concavity of the function f, we must examine:

- All points $x \in D(f)$ where the function f is continuous and for which exists $f''(x) = 0$.
- All points $x \in D(f)$ where the function f is continuous and in which $f''(x)$ does not exist.

04. Investigation of behaviour of functions

We can generalize the previous results.

$y = f(x)$, $x \in D(f)$, point $x_0 \in D(f)$, $n \in N$.

$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$, $f^{(n)}(x_0) \neq 0$.

- $n = 2k-1$, $k \in N$ (odd).
  - $f^{(n)}(x_0) > 0$. $\Rightarrow$ • f is increasing at the point x_0.
  - $f^{(n)}(x_0) < 0$. $\Rightarrow$ • f is decreasing at the point x_0.

  } $f(x_0)$ is not extremal.
- $n = 2k$, $k \in N$ (even).
  - $f^{(n)}(x_0) > 0$. $\Rightarrow$ • $f(x_0)$ is a strict local minimum.
  - $f^{(n)}(x_0) < 0$. $\Rightarrow$ • $f(x_0)$ is a strict local maximum.

$y = f(x)$, $x \in D(f)$, point $x_0 \in D(f)$, $n \in N$.

$f'(x_0) \in R$, $f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$, $f^{(n)}(x_0) \neq 0$.

- $n = 2k+1$, $k \in N$ (odd).
  - x_0 is the inflection point of the function f.
- $n = 2k$, $k \in N$ (even).
  - $f^{(n)}(x_0) > 0$. $\Rightarrow$ • f is strictly convex at the point x_0.
  - $f^{(n)}(x_0) < 0$. $\Rightarrow$ • f is strictly concave at the point x_0.

05. Behaviour of functions

Investigating the behaviour of the function f means determining:

- Domain $D(f)$, points and intervals of continuity and discontinuity.
- Evenness, oddness, periodicity, respectively other special properties.
- One-sided limits at discontinuity points, boundary points, and $\pm\infty$ points.
- Zero points; intervals on which f is positive and negative.
- f', stationary points, local and global extrema; intervals on which f is increasing, decreasing and constant.
- f'', inflection points; intervals on which f is convex and concave.
- Asymptotes without slope and asymptotes with slope.
- Codomain $H(f)$ and sketch the graph of the function.

  The graph usually gives us the most vivid idea of the behaviour of the function. During its construction, we use all the data found.

  Of course, many times they are insufficient, so we have to supplement them with appropriately chosen functional values.

05. Behaviour of functions

The behaviour of the function $f(x) = \frac{8(x-2)}{x^2} = \frac{8x-16}{x^2}$.

```
(%i1) f(x):=(8*x-16)/x^2;
```
(%o1) $f(x) := \frac{8x-16}{x^2}$

- $D(f) = R - \{0\} = (-\infty; 0) \cup (0; \infty)$.

  Using the command `denom` (denominator) we find out when the denominator is zero.

```
(%i3) fm:denom(f(x));solve(fm=0,x);
```
(fm) x^2

(%o3) $[x = 0]$

- f is not periodic, f is not even, f is not odd.
- f is continuous on intervals $(-\infty; 0)$, $(0; \infty)$, is discontinuous at the point 0.

05. Behaviour of functions

- $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} \frac{8x-16}{x^2} = \lim\limits_{x\to\pm\infty} \left(\frac{8}{x} - \frac{16}{x^2}\right) = \frac{8}{\pm\infty} - \frac{16}{\infty} = 0 - 0 = 0.$

```
(%i5) limit(f(x),x,minf);limit(f(x),x,inf);
(%o4) 0
(%o5) 0
```

- $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \frac{8(x-2)}{x^2} = \frac{-16}{0^+} = -\infty, \ \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \frac{8(x-2)}{x^2} = \frac{-16}{0^+} = -\infty.$

```
(%i7) limit(f(x),x,0,minus);limit(f(x),x,0,plus);
(%o6) -∞
(%o7) -∞
```

- The point $x = 0$ is an non-removable discontinuity point of the II. type.
- $x = 0$ is an asymptote without a slope.

05. Behaviour of functions

- $f(x) = \frac{8x-16}{x^2} = 0. \Leftrightarrow 8x - 16 = 0. \Leftrightarrow x = 2.$

  Using the command `num` (numerator) we find out when the numerator is zero.

```
(%i9)  fc:num(f(x));solve(fc=0,x);
(fc)   8x − 16
(%o9)  [x = 2]
```

- $f(2) = 0.$
- f is not defined at the point $x = 0$.
- The function f does not change the sign on intervals $(-\infty; 0)$, $(0; 2)$, $(2; \infty)$.
- Just select any point in the given intervals and verify its value (e.g. -1, 1, 3).

```
(%i13) f(2);f(-1);f(1);f(3);
(%o10) 0
(%o11) −24
(%o12) −8
(%o13) 8/9
```

05. Behaviour of functions

- $-1 \in (-\infty; 0)$, $f(-1) = -24 < 0$. $\Rightarrow$ • $f(x) < 0$ for $x \in (-\infty; 0)$.
- $1 \in (0; 2)$, $f(1) = -8 < 0$. $\Rightarrow$ • $f(x) < 0$ for $x \in (0; 2)$.
- $3 \in (2; \infty)$, $f(3) = \frac{8}{9} > 0$. $\Rightarrow$ • $f(x) > 0$ for $x \in (2; \infty)$.

- $f'(x) = \left[\frac{8x-16}{x^2}\right]' = \frac{8x^2-(8x-16)2x}{x^4} = \frac{32x-8x^2}{x^4} = \frac{32-8x}{x^3}$, $x \in R$, $x \neq 0$.

```
(%i15) f1(x):=diff(f(x),x,1)$
       ratsimp(f1(x));
```
(%o15) $-\frac{8x-32}{x^3}$

- $f'(x) = \frac{32-8x}{x^3} = 0. \Leftrightarrow 32 - 8x = 0. \Leftrightarrow x = 4.$

```
(%i16) solve(f1(x)=0,x);
```
(%o16) $[x = 4]$

05. Behaviour of functions

- f' is discontinuous at 0.

```
(%i18) f1m:denom(ratsimp(f1(x)));solve(f1m=0,x);
```

(f1m) x^3

(%o18) $[x = 0]$

- $f'(4) = 0$.
- f' is not defined at the point $x = 0$.
- The function f' does not change the sign on intervals $(-\infty; 0)$, $(0; 4)$, $(4; \infty)$.
- Just select any point in the given intervals and verify its value (e.g. -1, 1, 5).

```
(%i22) subst(4,x,f1(x));
       subst(-1,x,f1(x));subst(1,x,f1(x));subst(5,x,f1(x));
```

(%o19) 0

(%o20) -40

(%o21) 24

(%o22) $-\frac{8}{125}$

05. Behaviour of functions

- $-1 \in (-\infty; 0)$, $f'(-1) = -40 < 0$. $\Rightarrow$ • $f'(x) < 0$, f is decreasing for $x \in (-\infty; 0)$.
- $1 \in (0; 4)$, $f'(1) = 24 > 0$. $\Rightarrow$ • $f'(x) > 0$, f is increasing for $x \in (0; 4)$.
- $5 \in (4; \infty)$, $f'(5) = -\frac{8}{125} < 0$. $\Rightarrow$ • $f'(x) < 0$, f is decreasing for $x \in (4; \infty)$.

- f has a local maximum at the point $x = 4$ and also a global maximum $f(4) = 1$.

```
(%i23) f(4);
(%o23) 1
```

- f has neither a local nor a global minimum.
- $f''(x) = \left[\frac{32-8x}{x^3}\right]' = \frac{-8x^3-(32-8x)3x^2}{x^6} = \frac{16x^3-96x^2}{x^6} = \frac{16x-96}{x^4}$, $x \in R$, $x \neq 0$.

```
(%i25) f2(x):=diff(f(x),x,2)$ ratsimp(f2(x));
(%o25) (16x-96)/x^4
```

05. Behaviour of functions

- $f''(x) = \frac{16x-96}{x^4} = 0. \Leftrightarrow 16x - 96 = 0. \Leftrightarrow x = 6.$

```
(%i26) solve(f2(x)=0,x);
```
(%o26) $[x = 6]$

- f'' is discontinuous at 0.

```
(%i28) f2m:denom(ratsimp(f2(x)));solve(f2m=0,x);
```
(f2m) x^4

(%o28) $[x = 0]$

- $f''(6) = 0.$
- f'' is not defined at the point $x = 0$.
- The function f'' does not change the sign on intervals $(-\infty; 0)$, $(0; 6)$, $(6; \infty)$.
- Just select any point in the given intervals and verify its value (e.g. -1, 1, 7).

05. Behaviour of functions

```
(%i32) subst(6,x,f2(x));
       subst(-1,x,f2(x));subst(1,x,f2(x));subst(7,x,f2(x));
```

(%o29) 0
(%o30) -112
(%o31) -80
(%o32) $\frac{16}{2401}$

- $-1 \in (-\infty; 0)$, $f''(-1) = -112 < 0$. $\Rightarrow$ • $f''(x) < 0$, f is concave for $x \in (-\infty; 0)$.
- $1 \in (0; 6)$, $f''(1) = -80 < 0$. $\Rightarrow$ • $f''(x) < 0$, f is concave for $x \in (0; 6)$.
- $7 \in (6; \infty)$, $f''(7) = \frac{16}{2401} > 0$. $\Rightarrow$ • $f''(x) > 0$, f is convex for $x \in (6; \infty)$.

- The point $x = 6$ is the inflection point of the function f.

```
(%i33) f(6);
```

(%o33) $\frac{8}{9}$

05. Behaviour of functions

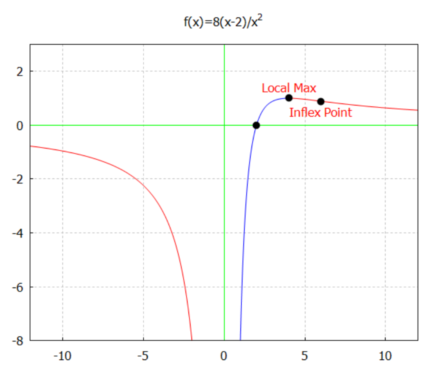
- $k = \lim\limits_{x\to\pm\infty} \frac{f(x)}{x} = \lim\limits_{x\to\pm\infty} \frac{8x-16}{x^3} = \lim\limits_{x\to\pm\infty} \left(\frac{8}{x^2} - \frac{16}{x^3}\right) = 0 - 0 = 0.$
- $q = \lim\limits_{x\to\pm\infty} [f(x) - kx] = \lim\limits_{x\to\pm\infty} [f(x) - 0\cdot x] = \lim\limits_{x\to\pm\infty} f(x) = 0.$
- $y = kx + q = 0\cdot x + 0 = 0$, i.e. $y = 0$ is an asymptote with a slope (horizontal).

```
(%i35) km:limit(f(x)/x,x,minf);
       qm:limit(f(x)-km*x,x,minf);
(km)   0
(qm)   0
(%i37) kp:limit(f(x)/x,x,inf);
       qp:limit(f(x)-kp*x,x,inf);
(kp)   0
(qp)   0
```

- $H(f) = (-\infty; 1\rangle.$

05. Behaviour of functions

(%i38)
```
draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[-12,12],yrange=[-8,3],title="f(x)=8(x-2)/x^2",
color=blue,explicit(f(x),x,0,4),
color=red,explicit(f(x),x,-12,0),explicit(f(x),x,4,12),
label(["Inflex Point",6,f(6)-.4],
["Local Max",4,f(4)+.4]),
color=green,
parametric(0,t,t,-8,3),
parametric(t,0,t,-12,12),
color=black,point_type=7,
points([[4,f(4)],
[6,f(6)],
[2,f(2)]]))$
```



04. Indefinite integral

Mathematical Analysis supported by wxMaxima

01. Basic Terms

All primitive functions to a given function $f(x)$, $x \in I$ on the interval I differ from each other by a constant and form the set $\{F(x) + c, c \in R\}$, where F is any of the primitive functions. This set is called **indefinite integral of the function f on the interval I** and is denoted

- $\int f(x)\,\mathrm{d}x = \{F(x) + c, x \in I, c \in R\} = F(x) + c,\ x \in I,\ c \in R.$

$f(x)$, $x \in I$ is continuous on the interval I.

$\Rightarrow$ • There exists $\int f(x)\,\mathrm{d}x$.

The `integrate` command is used to integrate in the wxMaxima.

(%i1) `'integrate(1/(1+x^2),x)`

(%o1) $\int \frac{1}{x^2+1}\,\mathrm{d}x$

01. Basic Terms

```
(%i1) f(x):=1/(1-x^2); integrate(f(x),x);
```
(%o1) $\frac{1}{1-x^2}$

(%o2) $\frac{\log(x+1)}{2} - \frac{\log(x-1)}{2}$

- Differentiation and integration are inverse operations on the interval I.

The function F is primitive to the function f on the interval I, $c \in R$.

For all $x \in I$ holds:

- $\left[\int f(x)\,dx\right]' = \left[F(x) + c\right]' = f(x)$.
- $\int F'(x)\,dx = \int f(x)\,dx = F(x) + c$.

```
(%i1) integrate(1/(1+x^2),x);
```
(%o1) $\operatorname{atan} x$

```
(%i2) diff(%,x);
```
(%o2) $\frac{1}{x^2+1}$

01. Basic Terms

- $\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \int \frac{[\sin x]'}{\sin x} \, dx = \ln |\sin x| + c, \; x \in R - \{k\pi, \; k \in Z\}, \; c \in R.$
- $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{-\sin x}{\cos x} \, dx = -\int \frac{[\cos x]'}{\cos x} \, dx = -\ln |\cos x| + c,$
  $x \in R - \{\frac{\pi}{2} + k\pi, \; k \in Z\}, \; c \in R.$
- $\int \sqrt[5]{x^3} \, dx = \int x^{\frac{3}{5}} \, dx = \frac{x^{\frac{3}{5}+1}}{\frac{3}{5}+1} + c = \frac{x^{\frac{8}{5}}}{\frac{8}{5}} + c = \frac{5}{8} x^{\frac{8}{5}} + c = \frac{5}{8} \sqrt[5]{x^8} + c, \; x \geq 0, \; c \in R.$

(%i1) `integrate(cot(x),x);`

(%o1) $\log(\sin x)$

(%i2) `integrate(tan(x),x);`

(%o2) $\log(\sec x)$

(%i3) `trigsimp(%);`

(%o3) $-\log(\cos x)$

(%i4) `integrate((x^3)^(1/5),x);`

(%o4) $\frac{5x^{5/8}}{8}$

01. Basic Terms

- $$\int |x|\,dx = \begin{cases} \int x\,dx = \frac{x^2}{2} + c = \frac{x\cdot x}{2} + c = \frac{x\,|x|}{2} + c & \text{for } x \geq 0, \\ \int (-x)\,dx = -\int x\,dx = -\frac{x^2}{2} + c = \frac{x\cdot(-x)}{2} + c = \frac{x\,|x|}{2} + c & \text{for } x < 0. \end{cases}$$

$\Rightarrow$ • $\int |x|\,dx = \frac{x\,|x|}{2} + c,\ x \in R,\ c \in R.$

```
(%i1) integrate(abs(x),x);
```
(%o1) $\frac{x\,|x|}{2}$

- $\int \frac{dx}{\sqrt{x^2-1}} = \ln\left|x + \sqrt{x^2-1}\right| + c,\ x \in (-\infty; -1) \cup (1; \infty).$ (tabular integral).

```
(%i1) integrate(1/sqrt(x^2-1),x);
```
(%o1) $\log\left(2\sqrt{x^2-1} + 2x\right)$

02. Methods of integration

- $\int \frac{dx}{\sqrt{x^2+1}} = \ln\left(x + \sqrt{x^2+1}\right) + c,\ x \in R.$ (tabular integral).

```
(%i1) integrate(1/sqrt(x^2+1),x);
(%o1) asinh x
```

- Both results are correct because the inverse hyperbolic sine function is defined by $y = \operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right),\ x \in R$ (see elementary functions).

Decomposition method.

Functions F, G are primitive to functions f, g on the interval I, $a, b \in R$, $|a| + |b| > 0$.

$\Rightarrow aF + bG$ is a primitive to the function $af + bg$ on the interval I and holds:

- $\int \left[af(x) + bg(x)\right] dx = a\int f(x)\, dx + b\int g(x)\, dx = aF(x) + bG(x) + c,\ x \in I,\ c \in R.$

- In practice, we write directly $\int \left[af(x) + bg(x)\right] dx = aF(x) + bG(x) + c.$

02. Methods of integration

- $\int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\, dx = \int \left[\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right] dx = \tan x - \cot x + c,$
  $x \in R,\ x \neq \frac{k\pi}{2},\ k \in Z,\ c \in R.$
- $\int \frac{(x-1)^2}{x}\, dx = \int \frac{x^2-2x+1}{x}\, dx = \int \left[x - 2 + \frac{1}{x}\right] dx = \frac{x^2}{2} - 2x + \ln|x| + c,\ x \in R - \{0\},\ c \in R.$
- $\int \left[2\cos x + x^3 + \frac{3}{x^2+1}\right] dx = 2\sin x + \frac{x^4}{4} + 3\arctan x + c,\ x \in R,\ c \in R.$

(%i1) `integrate(1/(sin(x)^2*cos(x)^2),x);`

(%o1) $\tan x - \frac{1}{\tan x}$

(%i2) `integrate((x-1)^2/x,x);`

(%o2) $\log x + \frac{x^2-4x}{2}$

(%i3) `integrate(2*cos(x)+x^3+3/(x^2+1),x);`

(%o3) $2\sin x + 3\,\mathrm{atan}\, x + \frac{x^4}{4}$

02. Methods of integration

Method per partes.

The functions u, v have continuous derivatives u', v' on the interval I.

$\Rightarrow$ • $\displaystyle\int u(x)\,v'(x)\,\mathrm{d}x = u(x)\,v(x) - \int u'(x)\,v(x)\,\mathrm{d}x,\ x \in I.$

• $[uv]' = u'v + uv'.$ $\Rightarrow$ • $\displaystyle uv = \int [uv]' = \int u'v + \int uv'.$ $\Rightarrow$ • $\displaystyle\int uv' = uv - \int u'v.$

- We can use the per partes method several times in a row, but we must be careful to stick to it they did not return to the original integral by reuse.
- The per partes method is used quite often. It is suitable for integrating functions

$$P(x)\,\mathrm{e}^{ax},\quad P(x)\cos ax,\quad P(x)\sin ax,\quad P(x)\ln Q(x),\quad P(x)\arctan Q(x),$$

where $P(x)$, $Q(x)$ are real polynomials, $a \in R$, $a \neq 0$.

- $\displaystyle\int \ln x\,\mathrm{d}x = \left[\begin{array}{ll|ll} u & = \ln x & u' & = \frac{1}{x} \\ v' & = 1 & v & = x \end{array}\right] = x\ln x - \int \mathrm{d}x = x\ln x - x + c,\ x \in (0;\infty),\ c \in R.$

02. Methods of integration

- $\int \arctan x \, dx = \left[\begin{array}{l|l} u' = 1 & u = x \\ v = \arctan x & v' = \frac{1}{1+x^2} \end{array} \right] = x \arctan x - \int \frac{x \, dx}{1+x^2} = x \arctan x - \frac{1}{2} \int \frac{0+2x}{1+x^2} \, dx$
  $= x \arctan x - \frac{1}{2} \ln \left| 1 + x^2 \right| + c = x \arctan x - \ln \sqrt{1 + x^2} + c, \ x \in R, \ c \in R.$
- $\int x \sin x \, dx = \left[\begin{array}{l|l} u = x & u' = 1 \\ v' = \sin x & v = -\cos x \end{array} \right] = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + c,$
  $x \in R, \ c \in R.$
- $\int x \cos x \, dx = \left[\begin{array}{l|l} u = x & u' = 1 \\ v' = \cos x & v = \sin x \end{array} \right] = x \sin x - \int \sin x \, dx = x \sin x + \cos x + c, \ x \in R, \ c \in R.$

```
(%i1) u:x; v:integrate(cos(x),x);
(u)   x
(v)   sin x
(%i3) u*v-integrate(v,x);
(%o3) x sin x + cos x
(%i4) integrate(x*cos(x),x);
(%o4) x sin x + cos x
```

02. Methods of integration

- $I_n = \int x^n e^x \, dx = \left[\begin{array}{l|l} u = x^n & u' = nx^{n-1} \\ v' = e^x & v = e^x \end{array} \right] = x^n e^x - n \int x^{n-1} e^x \, dx = x^n e^x - nI_{n-1}, \; n \in N.$

$\Rightarrow$
- $I_0 = \int x^0 e^x \, dx = \int e^x \, dx = e^x + c,$
- $I_1 = x e^x - I I_0 = x e^x - e^x + c,$
- $I_2 = x^2 e^x - 2 I I_1 = x^2 e^x - 2 \left[x e^x - e^x \right] + c = x^2 e^x - 2x e^x + 2 e^x + c,$
- $I_3 = x^3 e^x - 3 I I_2 = x^3 e^x - 3 \left[x^2 e^x - 2x e^x + 2 e^x \right] + c = x^3 e^x - 3x^2 e^x + 6x e^x - 6 e^x + c.$

```
(%i1) I(n,x):=integrate(x^n*exp(x),x)$
      I(0,x); I(1,x); I(2,x); I(3,x); I(4,x); I(5,x);
```

(%o2) e^x

(%o3) $(x-1)\, e^x$

(%o4) $(x^2 - 2x + 2)\, e^x$

(%o5) $(x^3 - 3x^2 + 6x - 6)\, e^x$

(%o6) $(x^4 - 4x^3 + 12x^2 - 24x + 24)\, e^x$

(%o7) $(x^5 - 5x^4 + 20x^3 - 60x^2 - 120x + 120)\, e^x$

02. Methods of integration

Substitution method.

A function F is primitive to the function f on the interval I,

$x = \varphi(t)$ has a derivative on the interval J, $\varphi(J) \subset I$.

$\Rightarrow F\big(\varphi(t)\big)$ is primitive to the function $f\big(\varphi(t)\big) \cdot \varphi'(t)$ on J and holds:

- $\displaystyle\int f\big(\varphi(t)\big) \cdot \varphi'(t)\,\mathrm{d}t = \int f(x)\,\mathrm{d}x = F(x) + c = F\big(\varphi(t)\big) + c,\ t \in J,\ c \in R.$

Sets I, J are intervals, $x = \varphi(t) : J \to I$ has a derivative $\varphi'(t) \neq 0$ on J,

a function $F(t)$ is primitive to $f\big(\varphi(t)\big) \cdot \varphi'(t)$ on J.

$\Rightarrow F\big(\varphi^{-1}(x)\big)$ is a primitive to the function $f(x)$ on interval I and holds:

- $\displaystyle\int f(x)\,\mathrm{d}x = \int f\big(\varphi(t)\big) \cdot \varphi'(t)\,\mathrm{d}t = F(t) + c = F\big(\varphi^{-1}(x)\big) + c,\ x \in I,\ c \in R.$

- In the first case we do not have to use inverse substitution,
  but in the second case we have to use the inverse substitution $t = \varphi^{-1}(x)$.

02. Methods of integration

- $\displaystyle\int \sin^3 t \cos t \, dt = \left[\begin{array}{r|l} \text{Subst. } x = \sin t & t \in R \\ dx = \cos t \, dt & x \in \langle -1; 1 \rangle \end{array} \right] = \int x^3 \, dx = \frac{x^{3+1}}{3+1} + c = \frac{\sin^4 t}{4} + c, \; t \in R, \; c \in R.$

- $\displaystyle\int \frac{f'(x)}{f(x)} \, dx = \left[\begin{array}{r} \text{Subst. } t = f(x) \\ dt = f'(x) \, dx \end{array} \right] = \int \frac{dt}{t} = \ln|t| + c = \ln|f(x)| + c, \; x \in D(f), \; c \in R.$

- $\displaystyle\int \frac{f'(t)}{f(t)} \, dt = \left[\begin{array}{r} \text{Subst. } x = f(t) \\ dx = f'(t) \, dt \end{array} \right] = \int \frac{dx}{x} = \ln|x| + c = \ln|f(t)| + c, \; t \in D(f), \; c \in R.$

- $\displaystyle\int \frac{x^2 \, dx}{x^3+1} = \left[\begin{array}{r|l} \text{Subst. } t = x^3 + 1 & x \in R \\ dt = 3x^2 \, dx & t \in R \end{array} \right] = \frac{1}{3} \int \frac{dt}{t} = \frac{1}{3} \ln|t| + c = \frac{1}{3} \ln\left|x^3 + 1\right| + c, \; x \in R, \; c \in R.$

- $\displaystyle\int \frac{x^2 \, dx}{x^6+1} = \left[\begin{array}{r|l} \text{Subst. } t = x^3 & x \in R \\ dt = 3x^2 \, dx & t \in R \end{array} \right] = \frac{1}{3} \int \frac{dt}{t^2+1} = \frac{1}{3} \arctan t + c = \frac{1}{3} \arctan x^3 + c, \; x \in R, \; c \in R.$

- $\displaystyle\int \frac{x^2 \, dx}{x^6-1} = \left[\begin{array}{r|l} \text{Subst. } t = x^3 & x \in R - \{\pm 1\} \\ dt = 3x^2 \, dx & t \in R - \{\pm 1\} \end{array} \right] = \frac{1}{3} \int \frac{dt}{t^2-1} = \frac{1}{3} \cdot \frac{1}{2} \ln\left|\frac{t-1}{t+1}\right| + c = \frac{1}{6} \ln\left|\frac{x^3-1}{x^3+1}\right|,$

  $x \in R - \{\pm 1\}, \; c \in R.$

02. Methods of integration

- $\displaystyle\int e^{5x}\,dx = \left[\begin{array}{r|l} \text{Subst. } 5x = t & x \in R \\ 5\,dx = dt & t \in R \end{array}\right]$

$$= \int e^{t}\,\frac{dt}{5} = \frac{1}{5}\int e^{t}\,dt = \frac{1}{5}\,e^{t} + c = \frac{1}{5}\,e^{5x} + c,\ x \in R,\ c \in R.$$

To perform a t-substitution in wxMaxima, we:

- Decide on a substitution and use `diff` to produce the differential dt (called `del(t)` in wxMaxima), then express dx in terms of dt using `solve`.
- Extract the resulting equation using `%[1]` and replace `del(x)` with its expression in terms of `del(t)` in the integrand.
- Use `subst` to transform the entire integrand in terms of t, then perform the integral, remembering that integrate expects only the coefficient of `del(t)`.
- Substitute the definition of t in terms of x into the resulting antiderivative.

02. Methods of integration

(%i1) `INTEGRAND:(%e^(5*x))*diff(x);`

(%o1) $e^{5x}\,\mathrm{del}(x)$

(%i2) `solve(diff(t)=diff(5*x),del(x));`

(%o2) $[\mathrm{del}(x) = \frac{\mathrm{del}(t)}{5}]$

(%i3) `%[1];`

(%o3) $\mathrm{del}(x) = \frac{\mathrm{del}(t)}{5}$

(%i5) `subst(rhs(%),del(x),INTEGRAND)$ subst(t,5*x,%);`

(%o5) $\frac{e^{t}\,\mathrm{del}(t)}{5}$

(%i6) `integrate(coeff(%,del(t)),t);`

(%o6) $\frac{e^{t}}{5}$

(%i7) `subst(5*x,u,%);`

(%o7) $\frac{e^{5x}}{5}$

(%i8) `integrate(%e^(5*x),x);`

(%o8) $\frac{e^{5x}}{5}$

02. Methods of integration

- $\displaystyle\int \frac{\ln x}{x}\,\mathrm{d}x = \left[\begin{array}{rr|l} \text{Subst.} & t = \ln x & x \in (0;\infty) \\ & \mathrm{d}t = \frac{\mathrm{d}x}{x} & t \in R \end{array}\right] = \int t\,\mathrm{d}t = \frac{t^2}{2} + c = \frac{\ln^2 x}{2} + c,\ x \in (0;\infty),\ c \in R.$

- $\displaystyle\int \frac{\ln x}{x}\,\mathrm{d}x = \left[\begin{array}{l|l} u = \ln x & u' = \frac{1}{x} \\ v' = \frac{1}{x} & v = \ln x \end{array}\right] = \ln^2 x - \int \frac{\ln x}{x}\,\mathrm{d}x.$

(Equation with the integral as the unknown.)

$\displaystyle\Rightarrow 2\int \frac{\ln x}{x}\,\mathrm{d}x = \ln^2 x + 2c. \Rightarrow$ • $\displaystyle\int \frac{\ln x}{x}\,\mathrm{d}x = \frac{\ln^2 x}{2} + c,\ x > 0,\ c \in R.$

A function $f(x)$ has on the interval I a primitive function $F(x)$, a real number $a, b \in R$, $a \neq 0$.

- $\displaystyle\int f(at+b)\,\mathrm{d}t = \left[\begin{array}{rr} \text{Subst.} & x = at+b \\ & \mathrm{d}x = a\,\mathrm{d}t \end{array}\right] = \int \frac{f(x)\,\mathrm{d}x}{a} = \frac{F(x)}{a} + c = \frac{F(at+b)}{a} + c.$

- $\displaystyle\int f(t+b)\,\mathrm{d}t = \left[\begin{array}{rr} \text{Subst.} & x = t+b \\ & \mathrm{d}x = \mathrm{d}t \end{array}\right] = \int f(x)\,\mathrm{d}x = F(x) + c = F(t+b) + c$ for $a = 1$.

- $\displaystyle\int f(-t)\,\mathrm{d}t = \left[\begin{array}{rr} \text{Subst.} & x = -t \\ & \mathrm{d}x = -\mathrm{d}t \end{array}\right] = -\int f(x)\,\mathrm{d}x = -F(x) + c = -F(-t) + c$ for $a = -1$.

05. Definite integral

Mathematical Analysis supported by wxMaxima

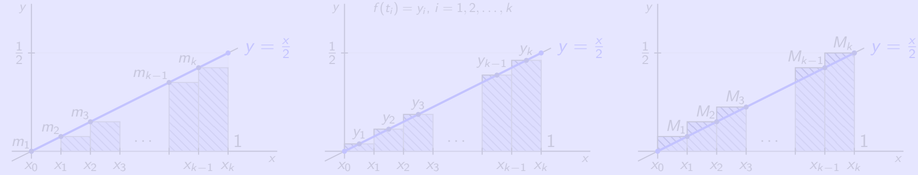
01. Basic Terms

- When investigating the Riemann integrable function f on the interval $\langle a; b\rangle$, we do not need all the dividing $D \in \mathfrak{D}_{\langle a;b\rangle}$.

  It is sufficient to restrict ourselves to **normal sequences** of dividing $\{D_k\}_{k=1}^{\infty} \subset \mathfrak{D}_{\langle a;b\rangle}$, i.e. if $\lim\limits_{k\to\infty} \mu(D_k) = 0$ holds.

  Then it holds for every choice of points T

$$\lim_{k\to\infty} S_T(f, D_k) = \int_a^b f(x)\,\mathrm{d}x.$$



$S_L(f, D_k) = \sum\limits_{i=1}^{k} m_i \cdot \Delta x_i = \sum\limits_{i=1}^{k} \frac{i-1}{2k\cdot k} = \frac{k-1}{4k}$ $\quad S_T(f, D_k) = \sum\limits_{i=1}^{k} f(t_i) \cdot \Delta x_i = \sum\limits_{i=1}^{k} \frac{2i-1}{4k\cdot k} = \frac{1}{4}$ $\quad S_U(f, D_k) = \sum\limits_{i=1}^{k} M_i \cdot \Delta x_i = \sum\limits_{i=1}^{k} \frac{i}{2k\cdot k} = \frac{k+1}{4k}$

$\int_0^1 \frac{x\,\mathrm{d}x}{2} = \frac{1}{4}$ (next page).

01. Basic Terms

- When investigating the Riemann integrable function f on the interval $\langle a; b\rangle$, we do not need all the dividing $D \in \mathfrak{D}_{\langle a;b\rangle}$.

  It is sufficient to restrict ourselves to **normal sequences** of dividing $\{D_k\}_{k=1}^{\infty} \subset \mathfrak{D}_{\langle a;b\rangle}$, i.e. if $\lim_{k\to\infty} \mu(D_k) = 0$ holds.

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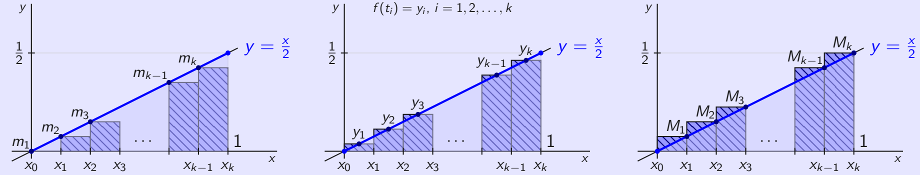

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$\int_0^1 \frac{x\,\mathrm{d}x}{2} = \frac{1}{4}$ (next page).

01. Basic Terms

$\int_0^1 \frac{x\,dx}{2} = \frac{1}{4}.$ The function $f(x) = \frac{x}{2}$, $x \in \langle 0;1\rangle$ is increasing, continuous, $f \in R_{\langle 0;1\rangle}$.

- A normal sequence of dividings $\{D_k\}_{k=1}^{\infty} \subset \mathfrak{D}_{\langle 0;1\rangle}$, while $D_k = \left\{\frac{i}{k}\right\}_{i=0}^{k}$ for $k \in N$.
- For $i = 1, 2, \ldots, k$ holds $\Delta x_i = \frac{1}{k}$, $m_i = f(x_{i-1}) = \frac{i-1}{2k}$, $M_i = f(x_i) = \frac{i}{2k}$.

$$S_L(f, D_k) = \sum_{i=1}^{k} m_i \cdot \Delta x_i = \sum_{i=1}^{k} \frac{i-1}{2k} \cdot \frac{1}{k} = \frac{0+1+\cdots+(k-1)}{2k^2} = \frac{\frac{(0+k-1)k}{2}}{2k^2} = \frac{k-1}{4k} = \frac{1}{4} - \frac{1}{4k}.$$

$$S_U(f, D_k) = \sum_{i=1}^{k} M_i \cdot \Delta x_i = \sum_{i=1}^{k} \frac{i}{2k} \cdot \frac{1}{k} = \frac{1+2+\cdots+k}{2k^2} = \frac{\frac{(1+k)k}{2}}{2k^2} = \frac{k+1}{4k} = \frac{1}{4} + \frac{1}{4k}.$$

$$\Rightarrow \bullet \int_0^1 \frac{x\,dx}{2} = \lim_{k\to\infty} S_L(f, D_k) = \lim_{k\to\infty} S_U(f, D_k) = \lim_{k\to\infty} \left(\frac{1}{4} \pm \frac{1}{4k}\right) = \frac{1}{4}.$$

- Let's choose $T = \{t_i\}_{i=1}^{k}$ as points centers of intervals $\langle x_{i-1}; x_i\rangle$, $i = 1, 2, \ldots, k$,
  i.e. $t_i = \frac{1}{2}\left(\frac{i-1}{k} + \frac{i}{2}\right) = \frac{2i-1}{2k}$, then $f(t_i) = \frac{2i-1}{4k}$ and holds

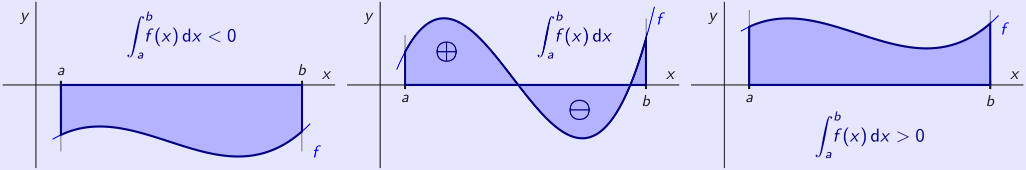
$$S_T(f, D_k) = \sum_{i=1}^{k} f(t_i) \cdot \Delta x_i = \sum_{i=1}^{k} \frac{2i-1}{4k} \cdot \frac{1}{k} = \frac{1+3+\cdots+(2k-1)}{4k^2} = \frac{\frac{(1+2k-1)k}{2}}{4k^2} = \frac{1}{4}.$$

$$\Rightarrow \bullet \int_0^1 \frac{x\,dx}{2} = \lim_{k\to\infty} S_T(f, D_k) = \lim_{k\to\infty} \frac{1}{4} = \frac{1}{4}.$$

02. Basic Properties

- Geometrically, it represents the Riemannian definite integral on the interval $\langle a; b\rangle$ the area of the curvilinear trapezoid determined by the function f and the interval $\langle a; b\rangle$.

  Below the x axis (i.e., if f is negative), this area is negative.



Functions $f, g \in R_{\langle a;b\rangle}$, a number $c \in R$.

$\Rightarrow cf, f+g, |f|, f^2, fg \in R_{\langle a;b\rangle}$ and holds:

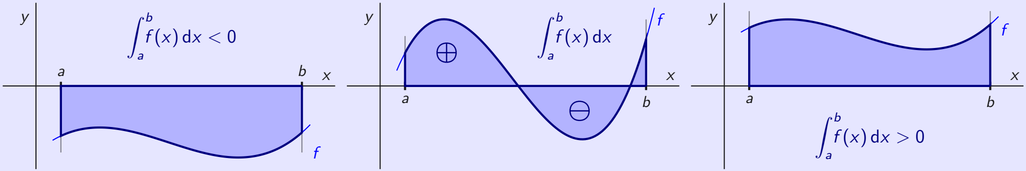
- $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx,$
- $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

If $\inf\{g(x), x\in\langle a;b\rangle\} > 0$, resp. $\sup\{g(x), x\in\langle a;b\rangle\} < 0$, then also $\frac{1}{g}, \frac{f}{g} \in R_{\langle a;b\rangle}$.

02. Basic Properties

- Geometrically, it represents the Riemannian definite integral on the interval $\langle a; b\rangle$ the area of the curvilinear trapezoid determined by the function f and the interval $\langle a; b\rangle$.

  Below the x axis (i.e., if f is negative), this area is negative.



Functions $f, g \in R_{\langle a;b\rangle}$, a number $c \in R$.

$\Rightarrow cf, f+g, |f|, f^2, fg \in R_{\langle a;b\rangle}$ and holds:

- $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx,$
- $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

If $\inf\{g(x), x \in \langle a; b\rangle\} > 0$, resp. $\sup\{g(x), x \in \langle a; b\rangle\} < 0$, then also $\frac{1}{g}, \frac{f}{g} \in R_{\langle a;b\rangle}$.

02. Basic Properties

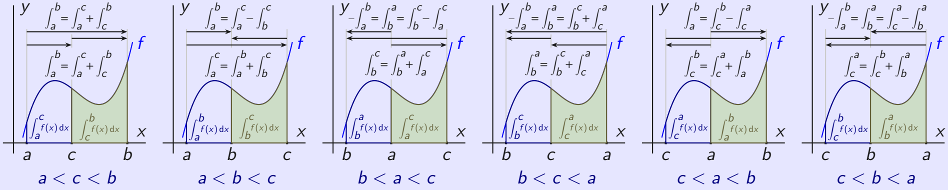
Functions $f, g \in R_{\langle a;b\rangle}$.

- $f(x) \geq 0$ for all $x \in \langle a; b\rangle$. $\Rightarrow$ • $\displaystyle\int_a^b f(x)\,\mathrm{d}x \geq 0.$
- $g(x) \geq f(x)$ for all $x \in \langle a; b\rangle$. $\Rightarrow$ • $\displaystyle\int_a^b g(x)\,\mathrm{d}x \geq \int_a^b f(x)\,\mathrm{d}x.$

Additivity of the integral.

A function $f \in R_I$, $I \subset R$ is a bounded interval, points $a, b, c \in I$ are arbitrary.

$\Rightarrow$ • $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$



We can clearly illustrate the additivity of the Riemann integral on vectors.

03. Methods of integration

Calculation of the Riemann integral (Newton-Leibniz formula).

A function $f \in R_{\langle a;b\rangle}$, the function F is a primitive function to the function f on $\langle a; b\rangle$.

$\Rightarrow$ • $\displaystyle\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = \Big[F(x)\Big]_a^b.$

- $\displaystyle\int_{-1}^{0} \frac{x}{2}\,\mathrm{d}x = \left[\frac{x^2}{2\cdot 2}\right]_{-1}^{0} = \left[\frac{x^2}{4}\right]_{-1}^{0} = \frac{0^2}{4} - \frac{(-1)^2}{4} = \frac{1}{4}.$
- $\displaystyle\int_{-1}^{1} x^2\,\mathrm{d}x = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1^2}{3} - \frac{(-1)^3}{3} = \frac{2}{3}.$
- $\displaystyle\int_{0}^{1} \frac{\mathrm{d}x}{\sqrt{x^2+1}} = \left[\ln\left(x + \sqrt{x^2+1}\right)\right]_{0}^{1} = \ln\left(1+\sqrt{2}\right) - \ln 1 = \ln\left(1+\sqrt{2}\right).$

```
(%i1) integrate(f(x),x,-1,1);
```

(%o1) $\displaystyle\int_{-1}^{1} f(x)\,\mathrm{d}x$

03. Methods of integration

Calculation of the Riemann integral (Newton-Leibniz formula).

A function $f \in R_{\langle a;b\rangle}$, the function F is a primitive function to the function f on $\langle a; b\rangle$.

$\Rightarrow$ • $\displaystyle\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = \Big[F(x)\Big]_a^b.$

- $\displaystyle\int_{-1}^{0} \frac{x}{2}\,\mathrm{d}x = \left[\frac{x^2}{2\cdot 2}\right]_{-1}^{0} = \left[\frac{x^2}{4}\right]_{-1}^{0} = \frac{0^2}{4} - \frac{(-1)^2}{4} = \frac{1}{4}.$
- $\displaystyle\int_{-1}^{1} x^2\,\mathrm{d}x = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1^2}{3} - \frac{(-1)^3}{3} = \frac{2}{3}.$
- $\displaystyle\int_{0}^{1} \frac{\mathrm{d}x}{\sqrt{x^2+1}} = \left[\ln\left(x + \sqrt{x^2+1}\right)\right]_0^1 = \ln\left(1+\sqrt{2}\right) - \ln 1 = \ln\left(1+\sqrt{2}\right).$

```
(%i1) integrate(f(x),x,-1,1);
```

(%o1) $\displaystyle\int_{-1}^{1} f(x)\,\mathrm{d}x$

03. Methods of integration

```
(%i2) f(x):=x^2$ F:integrate(f(x),x);
```

(F) $\frac{x^3}{3}$

```
(%i3) integrate(f(x),x,-1,1);
```

(%o3) $\frac{2}{3}$

```
(%i4) subst(1,x,F)-subst(-1,x,F);
```

(%o4) $\frac{2}{3}$

```
(%i5) float(subst(1,x,F)-subst(-1,x,F));
```

(%o5) 0.6666666666666666

```
(%i6) float(integrate(f(x),x,-1,1));
```

(%o6) 0.6666666666666666

```
(%i7) bfloat(integrate(f(x),x,-1,1));
```

(%o7) $6.666666666666667b-1$

03. Methods of integration

```
(%i2) f(x):=cos(x)*sin(x)$ F:integrate(f(x),x);
```

(F) $-\frac{\cos x^2}{2}$

```
(%i3) integrate(f(x),x,1,2);
```

(%o3) $\frac{\cos 1^2}{2} - \frac{\cos 2^2}{2}$

```
(%i4) subst(2,x,F)-subst(1,x,F);
```

(%o4) $\frac{\cos 1^2}{2} - \frac{\cos 2^2}{2}$

```
(%i5) float(integrate(f(x),x,1,2));
```

(%o5) 0.05937419607911741

```
(%i6) float(subst(2,x,F)-subst(1,x,F));
```

(%o6) 0.05937419607911741

```
(%i7) bfloat(subst(2,x,F)-subst(1,x,F));
```

(%o7) $5.937419607911738b-2$

03. Methods of integration

- $\displaystyle\int_{-1}^{1}\frac{dx}{x} = \Big[\ln|x|\Big]_{-1}^{1} = \ln 1 - \ln 1 = 0 \quad (?!?).$
- The function $\frac{1}{x}$ is not defined at the point 0.
- The function $\frac{1}{x}$ is not bounded on the intervals $\langle -1; 0)$ and $(0; 1\rangle$.
- In this sense we cannot calculate the integral.

```
(%i2) f(x):=1/x$ F:integrate(f(x),x);
(F)   -log x
(%i3) integrate(f(x),x,-1,1);
      Principal Value
(%o3) 0
(%i4) subst(1,x,F)-subst(-1,x,F);
(%o4) -log(-1)
```

03. Methods of integration

- Definite integrals are generally calculated using indefinite integrals.
- We can modify the per partes method and substitution methods and calculate the definite integral using them directly.

  After substitution, we do not need to return to the original variables.

Method per partes.

$$u, u', v, v' \in R_{\langle a;b\rangle}. \Rightarrow \bullet \int_a^b u(x)\, v'(x)\, dx = \Big[u(x)\, v(x)\Big]_a^b - \int_a^b u'(x)\, v(x)\, dx.$$

$$\int_0^{2\pi} x^2 \sin x \, dx = \left[\begin{array}{l|l} u = x^2 & u' = 2x \\ v' = \sin x & v = -\cos x \end{array}\right] = \Big[-x^2 \cos x\Big]_0^{2\pi} + \int_0^{2\pi} 2x \cos x \, dx$$

$$= \left[\begin{array}{l|l} u = 2x & u' = 2 \\ v' = \cos x & v = \sin x \end{array}\right] = \Big[-4\pi^2 \cdot 1 + 0^2 \cdot 1\Big] + \Big[2x \sin x\Big]_0^{2\pi} - \int_0^{2\pi} 2 \sin x \, dx$$

$$= -4\pi^2 + \Big[4\pi \cdot 0 - 2 \cdot 0 \cdot 0\Big] - \Big[-2\cos x\Big]_0^{2\pi} = -4\pi^2 - \Big[-2 \cdot 1 + 2 \cdot 1\Big] = -4\pi^2.$$

03. Methods of integration

Substitution method. $y = f(x)\colon I \to R,\ x = \varphi(t)\colon J \to R.$

f is continuous on I, φ' is continuous on J, $\varphi(J) \subset I$,
I is an interval with boundaries a, b, J is an interval with boundaries α, β, $\varphi(\alpha) = a$, $\varphi(\beta) = b$.
$\Rightarrow f(\varphi)\varphi' \in R_J$ and holds • $\displaystyle\int_a^b f(x)\,dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)\,dt.$ (We can use in both directions.)

- $$\int_{-1}^{1}\sqrt{1-x^2}\,dx \left[\begin{array}{r|l|r|c} \text{Subst. } x=\sin t & x\in\langle -1;1\rangle & 1=\sin\frac{\pi}{2} & \sqrt{1-x^2}=\sqrt{1-\sin^2 t}=\sqrt{\cos^2 t}=|\cos t|=\cos t \\ dx=\cos t\,dt & t\in\langle -\frac{\pi}{2};\frac{\pi}{2}\rangle & -1=\sin\left(-\frac{\pi}{2}\right) & \cos t\geq 0 \text{ pre všetky } t\in\langle -\frac{\pi}{2};\frac{\pi}{2}\rangle \end{array}\right]$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos 2t}{2}\,dt = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[1+\cos 2t]\,dt = \frac{1}{2}\left[t+\frac{\sin 2t}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2}+\frac{\sin\pi}{2}-\left(-\frac{\pi}{2}+\frac{\sin(-\pi)}{2}\right)\right] = \frac{1}{2}\left[\frac{\pi}{2}+0-\left(-\frac{\pi}{2}+0\right)\right] = \frac{1}{2}\cdot\pi = \frac{\pi}{2}.$$

- $$\int_{-1}^{2} t\sin(t^2+1)\,dt = \left[\begin{array}{r|l|l|l} \text{Subst. } x=t^2+1 & t\in\langle -1;0\rangle & x\in\langle 1;2\rangle & t=2 \mapsto x=5 \\ dx=2\,dt & t\in\langle 0;2\rangle & x\in\langle 1;5\rangle & t=-1\mapsto x=2 \end{array}\right] = \frac{1}{2}\int_2^5 \sin x\,dx$$

$$= \frac{1}{2}\Big[-\cos x\Big]_2^5 = \frac{1}{2}\Big[-\cos 5+\cos 2\Big] = \frac{\cos 2-\cos 5}{2}.$$

03. Methods of integration

Substitution method. $y = f(x)\colon I \to R,\ x = \varphi(t)\colon J \to R.$

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- $$\int_{-1}^{1} \sqrt{1-x^2}\,dx \left[\begin{array}{r|l|r|c} \text{Subst. } x = \sin t & x \in \langle -1;1\rangle & 1 = \sin\frac{\pi}{2} & \sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \sqrt{\cos^2 t} = |\cos t| = \cos t \\ dx = \cos t\,dt & t \in \langle -\frac{\pi}{2}; \frac{\pi}{2}\rangle & -1 = \sin(-\frac{\pi}{2}) & \cos t \geq 0 \text{ pre všetky } t \in \langle -\frac{\pi}{2}; \frac{\pi}{2}\rangle \end{array}\right]$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}\,dt = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [1+\cos 2t]\,dt = \frac{1}{2}\left[t + \frac{\sin 2t}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\sin\pi}{2} - \left(-\frac{\pi}{2} + \frac{\sin(-\pi)}{2}\right)\right] = \frac{1}{2}\left[\frac{\pi}{2} + 0 - \left(-\frac{\pi}{2} + 0\right)\right] = \frac{1}{2}\cdot\pi = \frac{\pi}{2}.$$

- $$\int_{-1}^{2} t\sin(t^2+1)\,dt = \left[\begin{array}{r|l|l|l} \text{Subst. } x = t^2+1 & t \in \langle -1;0\rangle & x \in \langle 1;2\rangle & t = 2 \mapsto x = 5 \\ dx = 2\,dt & t \in \langle 0;2\rangle & x \in \langle 1;5\rangle & t = -1 \mapsto x = 2 \end{array}\right] = \frac{1}{2}\int_2^5 \sin x\,dx$$

$$= \frac{1}{2}\Big[-\cos x\Big]_2^5 = \frac{1}{2}\Big[-\cos 5 + \cos 2\Big] = \frac{\cos 2 - \cos 5}{2}.$$

03. Methods of integration

Substitution method. $y=f(x)\colon I\to R,\ x=\varphi(t)\colon J\to R.$

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I is an interval with boundaries a, b, J is an interval with boundaries α, β, $\varphi(\alpha)=a$, $\varphi(\beta)=b$.
$\Rightarrow f(\varphi)\varphi' \in R_J$ and holds • $\displaystyle\int_a^b f(x)\,dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)\,dt.$ (We can use in both directions.)

- $$\int_{-1}^{1}\sqrt{1-x^2}\,dx \left[\begin{array}{r|l|r|c} \text{Subst. } x=\sin t & x\in\langle -1;1\rangle & 1=\sin\frac{\pi}{2} & \sqrt{1-x^2}=\sqrt{1-\sin^2 t}=\sqrt{\cos^2 t}=|\cos t|=\cos t \\ dx=\cos t\,dt & t\in\left\langle -\frac{\pi}{2};\frac{\pi}{2}\right\rangle & -1=\sin\left(-\frac{\pi}{2}\right) & \cos t\geq 0 \text{ pre všetky } t\in\left\langle -\frac{\pi}{2};\frac{\pi}{2}\right\rangle \end{array}\right]$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,dt=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos 2t}{2}\,dt=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[1+\cos 2t\right]dt=\frac{1}{2}\left[t+\frac{\sin 2t}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=\frac{1}{2}\left[\frac{\pi}{2}+\frac{\sin\pi}{2}-\left(-\frac{\pi}{2}+\frac{\sin(-\pi)}{2}\right)\right]=\frac{1}{2}\left[\frac{\pi}{2}+0-\left(-\frac{\pi}{2}+0\right)\right]=\frac{1}{2}\cdot\pi=\frac{\pi}{2}.$$

- $$\int_{-1}^{2} t\sin(t^2+1)\,dt=\left[\begin{array}{r|l|l|l} \text{Subst. } x=t^2+1 & t\in\langle -1;0\rangle & x\in\langle 1;2\rangle & t=2 \mapsto x=5 \\ dx=2\,dt & t\in\langle 0;2\rangle & x\in\langle 1;5\rangle & t=-1\mapsto x=2 \end{array}\right]=\frac{1}{2}\int_2^5\sin x\,dx$$

$$=\frac{1}{2}\Big[-\cos x\Big]_2^5=\frac{1}{2}\Big[-\cos 5+\cos 2\Big]=\frac{\cos 2-\cos 5}{2}.$$

04. Integrating even and odd functions

$a \in R,\ m, n \in N,\ m \neq n.$

- $$\int_a^{a+2\pi} \sin^2(nx)\,dx = \int_0^{2\pi} \sin^2(nx)\,dx = \int_0^{2\pi} \frac{1-\cos(2nx)}{2}\,dx = \left[\frac{x}{2} - \frac{\sin(2nx)}{2\cdot 2n}\right]_0^{2\pi}$$
  $$= \left[\frac{2\pi}{2} - \frac{\sin(2n\cdot 2\pi)}{4n} - \frac{0}{2} + \frac{\sin 0}{4n}\right] = \left[\pi - 0 - 0 + 0\right] = \pi.$$

- $$\int_a^{a+2\pi} \cos^2(nx)\,dx = \int_0^{2\pi} \cos^2(nx)\,dx = \int_0^{2\pi} \frac{1+\cos(2nx)}{2}\,dx = \left[\frac{x}{2} + \frac{\sin 2nx}{2\cdot 2n}\right]_0^{2\pi}$$
  $$= \left[\frac{2\pi}{2} + \frac{\sin(2n\cdot 2\pi)}{4n} - \frac{0}{2} - \frac{\sin 0}{4n}\right] = \left[\pi + 0 - 0 - 0\right] = \pi.$$

- $$\int_a^{a+2\pi} \sin(mx)\sin(nx)\,dx = \int_0^{2\pi} \sin(mx)\sin(nx)\,dx = \int_0^{2\pi} \frac{\cos(mx-nx)-\cos(mx+nx)}{2}\,dx$$
  $$= \left[\frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}\right]_0^{2\pi} = \left[\frac{\sin(m-n)2\pi}{2(m-n)} - \frac{\sin(m+n)2\pi}{2(m+n)} - \frac{\sin 0}{2(m-n)} + \frac{\sin 0}{2(m+n)}\right] = 0.$$

- $$\int_a^{a+2\pi} \sin(mx)\cos(nx)\,dx = \int_0^{2\pi} \sin(mx)\cos(nx)\,dx = \int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0.$$

04. Integrating even and odd functions

(%i1) `f(x,n):=sin(n*x)^2;`

(%o1) $f(x,n) : \sin(nx)^2$

(%i2) `integrate(f(x,n),x,0,2*%pi);`

(%o2) $-\frac{\sin(4\pi n)-4\pi n}{4n}$

(%i3) `integrate(f(x,n),x,a+0,a+2*%pi);`

(%o3) $\frac{\sin(2an)-2an}{4n} - \frac{\sin((2a+4\pi)n)+(-2a-4\pi)n}{4n}$

(%i4) `ratsimp(%o3);`

(%o4) $-\frac{\sin((2a+4\pi)n)-\sin(2an)-4\pi n}{4n}$

(%i5) `integrate(f(x,4),x,0,2*%pi);`

(%o5) π

(%i6) `integrate(f(x,4),x,a+0,a+2*%pi);`

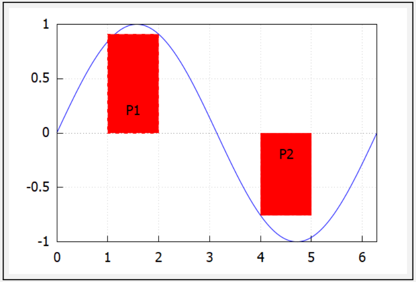
(%o6) $\frac{\sin(8a)-8a}{16} - \frac{\sin(8a)-8a-16\pi}{16}$

(%i7) `ratsimp(%);`

(%o8) π

04. Integrating even and odd functions

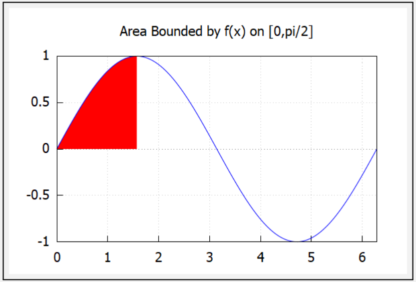
(%i1)
```
f(x):=sin(x)$
wxdraw2d(grid=true,xaxis=true,yaxis=true,
color=blue, explicit(f(x),x,0,2*%pi),border=false,
rectangle([1,0],[2,f(2)]),color=black,
label(["P1",1.5,0.2]),
rectangle([4,f(4)],[5,0]),color=black,
label(["P2",4.5,-0.2]));
```



04. Integrating even and odd functions

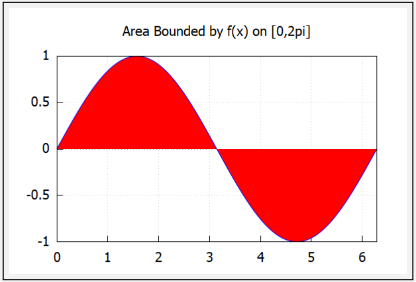
- Recall that $\int_a^b f(x)\,\mathrm{d}x$ gives us the area bounded between $f(x)$ and the x-axis.

```
(%i1) f(x):=sin(x)$
      wxdraw2d(grid=true,xrange=[0,2*%pi],yrange=[-1,1],
      xaxis=true,yaxis=true,
      title="Area Bounded by f(x) on [0,pi/2]",
      fill_color=red,filled_func=true,filled_func=f(x),
      explicit(0,x,0,%pi/2),filled_func=false,
      color=blue,explicit(f(x),x,0,2*%pi));
```



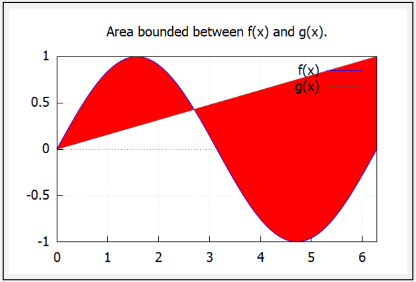
04. Integrating even and odd functions

```
(%i1) f(x):=sin(x)$
      wxdraw2d(grid=true,xrange=[0,2*%pi],yrange=[-1,1],
      xaxis=true,yaxis=true,
      title="Area Bounded by f(x) on [0,2pi]",
      fill_color=red,filled_func=true,filled_func=f(x),
      explicit(0,x,0,2*%pi),filled_func=false,
      color=blue,explicit(f(x),x,0,2*%pi));
```



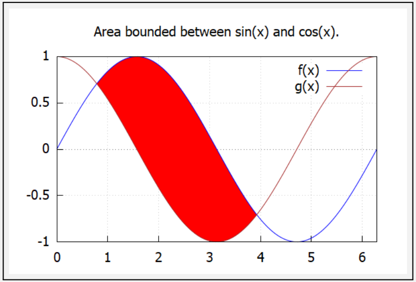
04. Integrating even and odd functions

(%i1)
```
f(x):=sin(x)$ g(x):=x/(2*%pi)$
wxdraw2d(grid=true,xaxis=true,yaxis=true,
xrange=[0,2*%pi],yrange=[-1,1],
title="Area bounded between f(x) and g(x).",
fill_color=red,filled_func=true,filled_func=f(x),
explicit(g(x),x,0,2*%pi),
filled_func=false,color=blue,key="f(x)",
explicit(f(x),x,0,2*%pi),color=brown,key="g(x)",
explicit(g(x),x,0,2*%pi));
```



04. Integrating even and odd functions

(%i1)
```
f(x):=sin(x)$ g(x):=cos(x)$
wxdraw2d(grid=true,xaxis=true,yaxis=true,
xrange=[0,2*%pi],yrange=[-1,1],
title="Area bounded between sin(x) and cos(x).",
fill_color=red,filled_func=true,filled_func=f(x),
explicit(g(x),x,%pi/4,5*%pi/4),
filled_func=false,color=blue,key="f(x)",
explicit(f(x),x,0,2*%pi),color=brown,key="g(x)",
explicit(g(x),x,0,2*%pi));
```



Thanks for your attention.

Mathematical Analysis supported by wxMaxima

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